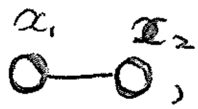
UNIVERSITY OF TECHNOLOGY SYDNEY

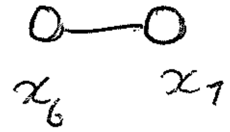
SCHOOL OF MATHEMATICAL AND PHYSICAL SCIENCES

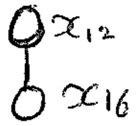
37457 Advanced Bayesian Methods

Practice Final Examination

SOLUTIONS

1. (a) The maximal cliques are 

, 

and 

, for example

and each contain 2 nodes.

(b) $p(x_1 | \text{rest}) = p(x_1 | \text{Markov blanket of } x_1)$

$= p(x_1 | x_2, x_5)$

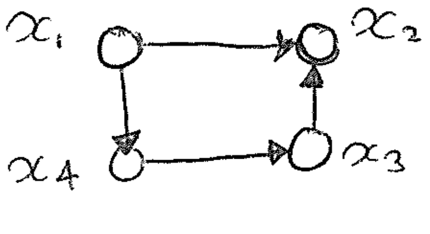
(c) Via the same argument used for (b),

i. $p(x_8 | \text{rest}) = p(x_8 | x_4, x_7, x_{12})$

ii. $p(x_{11} | \text{rest}) = p(x_{11} | x_7, x_{10}, x_{15}, x_{15})$

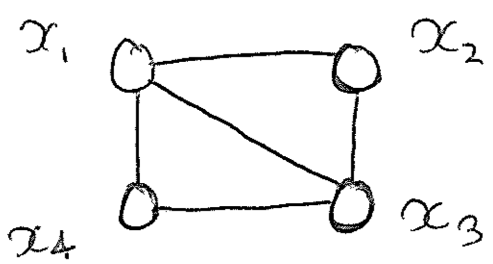
iii. $p(x_{13} | \text{rest}) = p(x_{13} | x_9, x_{14})$

(d) i. The smallest ancestral sub-graph containing $\{x_1, x_2, x_3, x_4\}$ is, trivially, the full DAG:



B

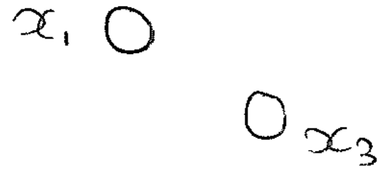
Moralising this DAG we get



Since $\{x_1, x_3\}$ separates $\{x_2\}$ and $\{x_4\}$ in this moral graph we can conclude that

$$x_2 \perp\!\!\!\perp x_4 \mid \{x_1, x_3\}$$

ii. The smallest ancestral sub-graph containing $\{x_1, x_3\}$ is



Moralising this DAG we get (trivially)

which is such that $\{x_1\}$ and $\{x_3\}$ are separated. Hence

$$x_1 \perp\!\!\!\perp x_3 \mid \phi$$ (ϕ is the empty set)

which simply means

$$x_1 \perp\!\!\!\perp x_3 .$$

(c)

(e)

$$p(x_1, x_2, x_3) \propto \psi_{12}(x_1, x_2)\, \psi_3(x_3)$$

$$= (x_1 + x_2 + 1)\, 6^{x_3}, \quad x_1, x_2, x_3 \in \{0, 1\}.$$

The normalising factor is:

$$\sum_{x_1=0}^{1} \sum_{x_2=0}^{1} \sum_{x_3=0}^{1} (x_1 + x_2 + 1)\, 6^{x_3}$$

$$= \left\{ \sum_{x_1=0}^{1} \sum_{x_2=0}^{1} (x_1 + x_2 + 1) \right\} \left\{ \sum_{x_3=0}^{1} 6^{x_3} \right\}$$

$$= \left\{ (0+0+1) + (0+1+1) + (1+0+1) + (1+1+1) \right\} \left\{ 6^0 + 6^1 \right\}$$

$$= (1 + 2 + 2 + 3)(1 + 6)$$

$$= 56$$

$$\Rightarrow p(x_1, x_2, x_3) = \begin{cases} \dfrac{(x_1 + x_2 + 1)\, 6^{x_3}}{56}, & x_1, x_2, x_3 \in \{0, 1\} \\ 0, & \text{otherwise} \end{cases}$$

2.

Ⓓ

(a) $\{\gamma_1, \gamma_2, \gamma_3\}$

(b) $\{\beta_0, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3, \mathbf{y}\}$

(c) $p(\sigma^2 | \text{rest}) = p(\sigma^2 | \text{Markov blanket of } \sigma^2)$

$$= p(\sigma^2 | \beta_0, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3, \mathbf{y})$$

$$\propto p(\mathbf{y}, \beta_0, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3, \sigma^2)$$

$$\propto p(\mathbf{y} | \beta_0, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3, \sigma^2)\, p(\sigma^2).$$

We are given $p(\mathbf{y} | \beta_0, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3, \sigma^2)$ and

$\sigma^2 \sim \text{Inverse-Gamma}(\sigma^2)$

$$\Rightarrow \quad p(\sigma^2) = \frac{(0.01)^{0.01}}{\Gamma(0.01)} (\sigma^2)^{-0.01-1} e^{-0.01/\sigma^2}, \quad \sigma^2 > 0$$

Ignoring multiplicative factors that don't involve σ^2:

$$p(\sigma^2 | \text{rest}) \propto \prod_{i=1}^{n} \left[(\sigma^2)^{-1/2} \exp\left\{ \frac{-(y_i - \beta_0 - \beta_1\gamma_1 x_{1i} - \beta_2\gamma_2 x_{2i} - \beta_3\gamma_3 x_{3i})^2}{2\sigma^2} \right\} \right]$$

$$\times (\sigma^2)^{-0.01-1} e^{-0.01/\sigma^2}, \quad \sigma^2 > 0$$

Ⓔ

$$\Rightarrow \quad p(\sigma^2|\text{rest}) \propto (\sigma^2)^{-(\frac{n}{2}+0.01)-1}$$

$$\times\, e^{-\frac{0.01+\frac{1}{2}\sum_{i=1}^{n}(y_i-\beta_0-\beta_1\gamma_1x_{1i}-\beta_2\gamma_2x_{2i}-\beta_3\gamma_3x_{3i})^2}{\sigma^2}}$$

which is proportional to the

Inverse Gamma density function with

shape parameter $A = \frac{n}{2}+0.01$

and scale parameter $B = 0.01+\frac{1}{2}\sum_{i=1}^{n}(y_i-\beta_0-\beta_1\gamma_1x_{1i}\cdots-\beta_3\gamma_3x_{3i})^2$.

Hence

$$\sigma^2|\text{rest} \sim \text{Inverse-Gamma}\left(0.01+\frac{n}{2},\, 0.01+\frac{1}{2}\sum_{i=1}^{n}(y_i-\beta_0-\beta_1\gamma_1x_{1i}\cdots-\beta_3\gamma_3x_{3i})^2\right).$$

(d) $$p(\rho|\text{rest}) = p(\rho|\gamma_1,\gamma_2,\gamma_3)$$

$$\propto p(\gamma_1,\gamma_2,\gamma_3|\rho)\,p(\rho)$$

$$\propto \rho^{\gamma_1+\gamma_2+\gamma_3}(1-\rho)^{3-(\gamma_1+\gamma_2+\gamma_3)}$$

$$\times\, \rho^{24}(1-\rho)^{39}$$

F

$\Rightarrow$

$$p(p|\text{rest}) \propto p^{(25+\gamma_1+\gamma_2+\gamma_3)-1}(1-p)^{43-(\gamma_1+\gamma_2+\gamma_3)-1}$$

$$\Rightarrow \quad p|\text{rest} \sim \text{Beta}\left(25+\sum_{j=1}^{3}\gamma_j,\ 43-\sum_{j=1}^{3}\gamma_j\right)$$

3.

(a)

$$p(\theta|x) \propto p(\theta, x)$$

$$= p(x|\theta)\,p(\theta)$$

$$\propto \prod_{i=1}^{10}\left(\theta e^{-\theta x_i}\right) . \theta^{39} e^{-5\theta}$$

$$= \theta^{49} e^{-\theta(5+\sum_{i=1}^{10} x_i)}$$

$$= \theta^{50-1} e^{-(5+\sum_{i=1}^{10} x_i)\theta}, \quad \theta > 0$$

$\Rightarrow$ $p(\theta|x)$ is the Gamma$(50, 5+\sum_{i=1}^{10} x_i)$ density function

$$\Rightarrow p(\theta|x) = \frac{(5+\sum_{i=1}^{10} x_i)^{50}}{\Gamma(50)} \theta^{49} e^{-(5+\sum_{i=1}^{10} x_i)\theta}, \quad \theta>0$$

(b) Using the result for the mean of the Gamma distribution given in the POSSIBLY USEFUL INFORMATION sheet:

(H)

$$\hat{\Theta}_{Bayes} = E(\Theta|x) = \frac{50}{5 + \sum_{i=1}^{10} x_i}.$$

(c) $\bar{x} = 9.96 \Rightarrow \sum_{i=1}^{10} x_i = 10 \times 9.96$
$= 99.6$

Hence

$$\hat{\Theta}_{Boyes} = \frac{50}{104.6} = \frac{500}{1046}.$$

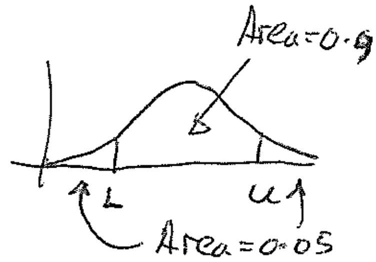
(d) Based on Carla's observed data

$p(\Theta|x)$ is the

Gamma$(50, 104.6)$ density function.

A 90% credible interval for Θ is

(L, u) where



$0.05 = P(G \leq L)$ and $0.95 = P(G \leq u)$

where $G \sim \text{Gamma}(50, 104.6)$

i.e. $0.05 = F(L)$ and $0.95 = F(u)$

where F is the cumulative distⁿ function of G

$\Leftrightarrow \quad L = Q(0.05)$ and $u = Q(0.95)$.

Carla should obtain

quantile_gamma(0.05, 50, 104.6)

and quantile_gamma(0.95, 50, 104.6)

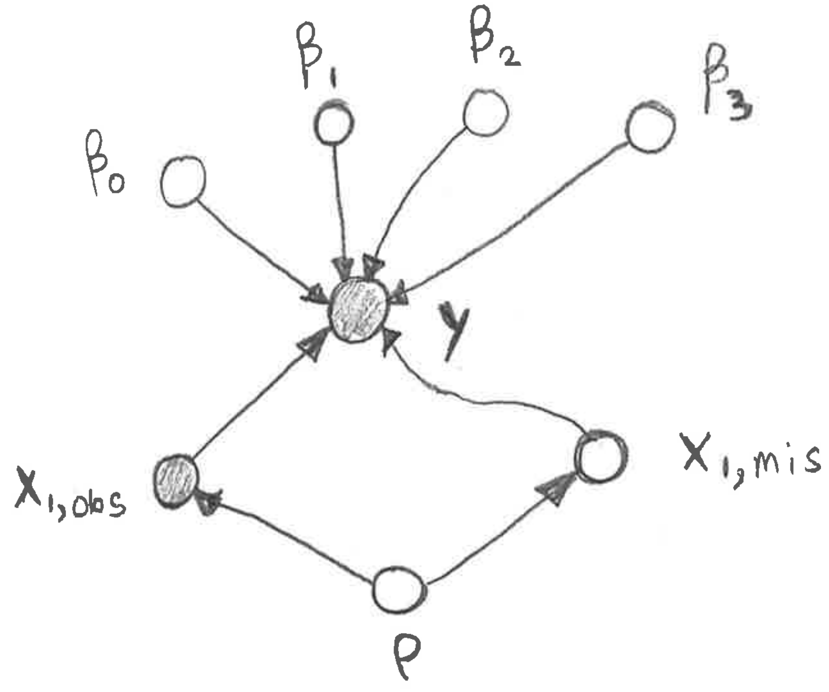
using SASQWATCH. These answers are the lower and upper limits, respectively, of the required 90% credible interval.

(e) Since, according to the density fⁿ $p(\theta|x)$,

$$0.9 = P(L < \theta < u)$$
$$= P\left(\frac{1}{u} < \frac{1}{\theta} < \frac{1}{L}\right)$$

the interval $\left(\frac{1}{u}, \frac{1}{L}\right)$ is a 90% credible interval for $1/\theta$.

4. (a)



(b) MEMORANDUM

From: Annika Chadwick

To: Leroy Meng

Re: Bayesian analysis of your lung diseuse study data

The Bayesian Poisson regression model (which accounted for missingness in the smoking data) revealed that smoking and dietary saturated fat have statistically signficant positive impacts on the mean response (number of carcinogen-DNA complexes in the lungs). There was no gender effect.

(c)