UNIVERSITY OF TECHNOLOGY SYDNEY

School of Mathematical and Physical Sciences

37457 Advanced Bayesian Methods

LABORATORY 1

Due time and date: 2:55pm, Friday 26th August, 2022.

Submission method: E-mail messages to Professor Wand (`matt.wand@uts.edu.au`).

NOTE: A formal help session has not been organised for this assessment task. If you require assistance and would like to set up a consultation time then please e-mail Professor Wand via `matt.wand@uts.edu.au`.

Goals: The goals of Laboratory 1 are:

- Learn some of the basics of the R statistical computing environment.

- See how probabilistic graphical model calculations can be carried out using the rjags packages within R. This will be important for the second topic of 37457 Advanced Bayesian Inference on Bayesian statistical inference.

1. **Downloading Required Files**

   (a) Choice of web browser: the use of `Mozilla Firefox` rather than `Microsoft Explorer` is recommended for all laboratories in 37457 Advanced Bayesian Inference since the latter web-browser has a tendency to change filenames (for example, `Microsoft Explorer` might add a `.txt` extension) which can cause problems later in this laboratory.

   (b) Decide on a folder in which you will store the files for this laboratory. It is best if you put them all in the same folder. A simple option is to put them on the `Desktop` folder, but please take note of where you store them for later.

   (c) Open `Mozilla Firefox` (assuming that this web browser is available on your computer – otherwise use another browser) and go to the go to the web-page:
   `http://matt-wand.utsacademics.info/37457.html`

   (d) Scroll down to the heading **Files for Laboratory 1** and download each of the files underneath this heading into the same folder. There are nine files in total with names:

   ```
   mitsub.R                 JessicaTheDAGwithData.R
   writeModel.r             NathanielTheDAGwithData.R
   BobTheDAG.R              summMCMC.r
   BobTheDAGwithData.R      ctsNathanielTheDAGwithData.R
   ```

   (e) The remainder of this laboratory assumes that these the filenames are **exactly** as listed here. If the downloading process corrupts the filenames in any way then you need to change the filenames to be these.

2. **Some R Basic Operations**

The following commands illustrate basic use of R and is intended only for those participants who are unfamiliar with R. If you are an experienced R user then you will probably want to skip this part.

(a) Start an R session. If you need any help with starting R then please ask the lecturer for help.

(b) Type `x <- 4` and hit `Enter`. From now on the line to type will be given (hitting `Enter` is assumed).

(c) Enter the commands:

```
y <- 7
sin(x*y)*sqrt(log(3*x+y))
```

(d) You have just computed $\sin(4 \times 7)\sqrt{\log(3 \times 4 + 7)}$ and should get the answer: 0.4648572

(e) It is useful to know that long commands in R can be recalled using the *up arrow* key. Suppose that we wanted to change the previous computation to that with $x = 18$ and $y = 2$. Then enter the command:

```
x <- 18 ; y <- 2
```

(f) Hit the *up arrow* key a couple of times until `sin(x*y)*sqrt(log(3*x+y))` reappears next to the prompt.

(g) Then hit `Enter` to perform this calculation again with the new x and y values. You should get the answer: -1.989834

(h) Enter the commands:

```
A <- cbind(c(6,-2),c(3,11))
A
```

The last command should print the 2×2 matrix A to the screen with columns $(6, -2)$ and $(3, 11)$.

(i) Enter the commands:

```
b <- c(4,-13)
A%*%b
```

You have just done the matrix multiplication

$$\begin{bmatrix} 6 & 3 \\ -2 & 11 \end{bmatrix}\begin{bmatrix} 4 \\ -13 \end{bmatrix} = \begin{bmatrix} -15 \\ -151 \end{bmatrix}.$$

(j) Enter the commands:

(**IMPORTANT**: Note that the `"l"` below is lower-case L and not the number 'one'.)

```
x <- seq(0,1,length=201)
y <- sin(2*pi*x^3)
plot(x,y,type="l",col="magenta")
```

You have just plotted the function $y = \sin(2\pi x^3)$ over the range $0 \le x \le 1$.

(k) Type `q()` to exit R. (Type `No` or `n` to the 'Save workspace image?' question.)

3. **Running R via scripts**

When several R (or, possibly, several thousand) R commands need to be run then it is better to use a *script*, which is simply a text file containing the commands.

(a) Open an R session and make sure that the current working directory corresponds to the directory/folder in which `mitsub.R` resides (i.e. the folder where you downloaded the files at the start of the laboratory).

(b) Inside R, issue the command `source("mitsub.R")`

This will run all several R commands listed in the file `mitsub.R`

(c) If successful, the previous step should lead to some graphics involving age and price of 35 Mitsubishi cars and a simple linear regression analysis being performed and summarised. If this doesn't happen then please ask for help from the lecturer.

(d) Open `mitsub.R` using your favourite editor. **However, because of the non-standard extension on the filename Windows is likely to ask you to choose the editor. A safe choice (if using Windows) is WordPad. The editor NotePad is usually not a good choice for cross-system text files.** Quickly read through the content of `mitsub.R` to see the R commands that you have just run.

WARNING: The remainder of Laboratory 1 depends on modern software packages that have become known as "Bayesian inference engines". Whilst they are extremely powerful, they are still being developed by various academic (rather than corporate) groups and are prone to occasional breakdown. Therefore, it is difficult to guarantee that each of the following examples will run on a random computer without a problem.

4. **Using the rjags package to do probabilistic graph calculations**

   Consider:

   ### Bob the DAG

   and suppose that the probability functions corresponding to Bob the DAG are:

   $$p(x_2) = \begin{cases} 1 - \wp, & x_2 = 0 \\ \wp, & x_2 = 1 \end{cases}$$

   $$= \wp^{x_2}(1 - \wp)^{1-x_2}, \quad x_2 = 0, 1,$$

   and

   $$p(x_1|x_2) = \begin{cases} 1 - f(x_2), & x_1 = 0 \\ f(x_2), & x_1 = 1 \end{cases}$$

   $$= \{f(x_2)\}^{x_1}\{1 - f(x_2)\}^{1-x_1}, \quad x_1, x_2 = 0, 1.$$

   In terms of (fairly) standard statistical distributional notation we have:

   $$x_2 \sim \text{Bernoulli}(\wp)$$
   $$x_1|x_2 \sim \text{Bernoulli}(f(x_2))$$

   The form of $f(x_2)$ and value of $\wp$ you should use for this exercise correspond to their values in your version of Assignment 2. In case these have slipped your mind, please get them from the list in the Appendix (last page).

   (a) If you not already have an active R session then start one and make sure that the current working directory corresponds to the directory/folder `BobTheDAG.R` re-side.s

   (b) This part of the laboratory assumes that the rjags package is correctly installed on the computer that you are using.

   (c) Type `source("BobTheDAG.R")`. If you get tables and graphics showing that the Monte Carlo answers of rjags approximately match the exact answers then all is well. **The results are best viewed with the graphics window made large.** If you do not know how to do this then please let the lecturer know and he will attempt to resolve the problem.

   (d) Copy `BobTheDAG.R` to a new file named `myBobTheDAG.R` and open the new file in your favourite editor (a reminder that a safe bet if using `Windows` is `WordPad`).

   (e) Modify the code to match your versions of $\wp$ and $f(x_2)$ from Assignment 2. As a reminder, your versions of $\wp$ and $f(x_2)$ are in the appendix of this laboratory. This requires the following steps which will be explained by example (but note

that you don't use the numbers in this example but those against your name in the appendix). Suppose that the new $\wp$ and $f(x_2)$ values are:

$$\wp = 0.77 \quad \text{and} \quad f(x_2) = \frac{37\,x_2 + 71}{47\,x_2 + 89}.$$

Then

i. the line:

`probx2eq1 <- 0.61` should be changed to `probx2eq1 <- 0.77`

ii. the line:

`probx1eq1CondOnx2 <- (1*x2 + 2)/(3*x2 + 7)` should be changed to `probx1eq1CondOnx2 <- (37*x2 + 71)/(47*x2 + 89)`

iii. the lines:

```
numerCoefx2 <- 1 ; numerConst <- 2
denomCoefx2 <- 3 ; denomConst <- 7
```

should be changed to:

```
numerCoefx2 <- 37 ; numerConst <- 71
denomCoefx2 <- 47 ; denomConst <- 89
```

Note, again, that you do not use these numbers but those against your name in the Appendix (last page) of this laboratory.

(f) Type `source("myBobTheDAG.R")`. If your modifications and mathematics in 3(a) are correct then you should tables and graphics showing that the Monte Carlo answers of rjags approximately match the exact answers. Continue until this happens.

(g) When the results match approximately then increase the `nIter` from ten thousand to one million and see if the accuracy of Monte Carlo improves. **NOTE: If the computer you are using is particularly old then one million iterations may take too long and a lower numbers should be considered instead.**

(h) If you have time then study the code in and around the `BobTheDAGModel` specification to learn more about how rjags works.

(i) E-mail your final version of `myBobTheDAG.R` to `matt.wand@uts.edu.au` (not right now, but before the deadline) so that it can be checked and credit for proper completion of Laboratory 1 awarded (but please note that you have not quite finished Laboratory 1 yet!).

Congratulations! You have now completed half of the Laboratory 1 assessment.

'Big f@#*ing deal' you think. I can get a nice exact answer to this problem with a few minutes of simple algebraic manipulations, while rjags takes about the same amount of fiddling around on the computer and can only get the answer approximately!

You may think that now – but by Week 6 we will see rjags effortlessly (if not quickly) taking care of the required calculations for models so complicated that the exact calculations will make you weep just thinking about them.

5. **Extension to probabilistic graphs with observed nodes ("data")**

   Now consider:

   Bob the DAG with data

   and suppose that the probability functions corresponding to Bob the DAG are:

   $$p(x_2) = \begin{cases} 1 - \wp, & x_2 = 0 \\ \wp, & x_2 = 1 \end{cases}$$

   $$= \wp^{x_2} (1 - \wp)^{1 - x_2}$$

   and

   $$p(x_1|x_2) = \begin{cases} 1 - f(x_2), & x_1 = 0 \\ f(x_2), & x_1 = 1 \end{cases}$$

   $$= \{f(x_2)\}^{x_1} \{1 - f(x_2)\}^{1 - x_1}.$$

   Let

   $$\overset{\circ}{x}_1 = \text{observed value of } x_1.$$

   (a) Consider the posterior probability function

   $$p(x_2|\overset{\circ}{x}_1)$$

   that you derived in Question 5 of Assignment 2. You should use $\wp$ and $f(x_2)$ values given in the Appendix.

   Section 2.8 of the notes *Graph Theory and Statistics* uses

   $$\wp = 0.61, \quad f(x_2) = \frac{x_2 + 2}{3\,x_2 + 7}.$$

   and this leads to the following table:

   $p(x_2|\overset{\circ}{x}_1)$ values:

	$x_2 = 0$	$x_2 = 1$
$\overset{\circ}{x}_1 = 0$	0.3948	0.6052
$\overset{\circ}{x}_1 = 1$	0.3785	0.6215

   (b) Type `source("BobTheDAGwithData.R")`. If you get tables and graphics showing that the Monte Carlo answers of **rjags** approximately match the exact answers then all is well. **The results are best viewed when the graphics window is made large.** If you do not know how to do this then please let the lecturer know and he will attempt to resolve the problem.

(c) Copy `BobTheDAGwithData.R` to a new file named `myBobTheDAGwithData.R` and open the new file in a text editor such as (if using `Windows`) `WordPad` (avoid `NotePad`).

(d) Modify the code to match your versions of $\wp$ and $f(x_2)$ This involves making changes analogous to those done earlier to the file `BobTheDAG.R` (described in Step 4(d)) using the values next to your name in the Appendix.

(e) Type `source("myBobTheDAGwithData.R")`. If your modifications and mathematics are correct then you should get tables and graphics showing that the Monte Carlo answers of rjags approximately match the exact answers. Continue until this happens.

(f) Edit `myBobTheDAGwithData.R` and change

```
x1Value <- 0
```

to

```
x1Value <- 1
```

and re-run the script. This corresponds to the data being $\overset{\circ}{x}_1 = 1$ rather than $\overset{\circ}{x}_1 = 0$.

(g) Edit `myBobTheDAGwithData.R` and change the `nIter` value to one million and see if the Monte Carlo accuracy improves.

(h) If you have time then study the code in and around the `BobTheDAGModel` specification to learn more about how rjags works.

(i) E-mail your final version of `myBobTheDAGwithData.R` (now or before the deadline) to `matt.wand@uts.edu.au` so that it can be checked and credit for proper completion of Laboratory 1 awarded.

This is the end of the formal part of Laboratory 1, in that all of the above must be completed to receive credit for its completion. The remaining pages contain some additional exercises involving *Jessica the DAG* and *Nathaniel the DAG*, which you may wish to do for further reinforcement of themes of Laboratory 1.

♦   ♦   ♦

6. Consider

Jessica the DAG with data
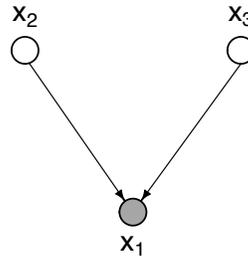
Let $p(x_2)$ and $p(x_3)$ be defined by:

x_2	1	2	3	4	5
$p(x_2)$	0.35	0.10	0.05	0.20	0.30

and

x_3	1	2	3	4	5
$p(x_3)$	0.28	0.19	0.09	0.11	0.33

Then set the conditional probability function on Jessica the DAG to be:

$$p(x_1|x_2, x_3) = \frac{\psi_{123}(x_1, x_2, x_3)}{\sum_{x_1'=1}^{5} \psi_{123}(x_1', x_2, x_3)}.$$

where the potential function $\psi_{123}(x_1, x_2, x_3)$ is defined to be

$$\psi_{123}(x_1, x_2, x_3) = \sqrt{\cosh(x_1 + 1.6\,x_2) + 0.4\,x_3}, \quad x_1, x_2, x_3 \in \{1, 2, 3, 4, 5\}.$$

The goal here is to obtain the posterior probability functions

$$p(x_2|\overset{\circ}{x}_1) \quad \text{and} \quad p(x_3|\overset{\circ}{x}_1)$$

where $\overset{\circ}{x}_1$ denotes the observed value of x_1 (i.e. "the data"). The first of these involves the mathematical steps:

$$
\begin{aligned}
p(x_2|x_1) &= \frac{p(x_1, x_2)}{p(x_1)} \\[2mm]
&= \frac{\sum_{x_3=1}^{5} p(x_1, x_2, x_3)}{\sum_{x_3=1}^{5} \sum_{x_2=1}^{5} p(x_1, x_2, x_3)} \\[2mm]
&= \frac{\sum_{x_3=1}^{5} p(x_1|x_2, x_3)\, p(x_2)\, p(x_3)}{\sum_{x_3=1}^{5} \sum_{x_2=1}^{5} p(x_1|x_2, x_3)\, p(x_2)\, p(x_3)} \\[2mm]
&= \frac{\displaystyle\sum_{x_3=1}^{5} \left\{ \frac{\psi_{123}(x_1, x_2, x_3)}{\sum_{x_1'=1}^{5} \psi_{123}(x_1', x_2, x_3)} \right\} p(x_2)\, p(x_3)}{\displaystyle\sum_{x_3=1}^{5} \sum_{x_2=1}^{5} \left\{ \frac{\psi_{123}(x_1, x_2, x_3)}{\sum_{x_1'=1}^{5} \psi_{123}(x_1', x_2, x_3)} \right\} p(x_2)\, p(x_3)}
\end{aligned}
$$

So setting $x_1 = \overset{\circ}{x}_1$ in this expression allows for "brute force" calculation of the posterior probability function $p(x_2|\overset{\circ}{x}_1)$.

The calculations for $p(x_3|\overset{\circ}{x}_1)$ are analogous.

(a) Open an R session and make sure that the current working directory corresponds to the directory/folder in which `JessicaTheDAGwithData.R` resides.

(b) This laboratory assumes that the rjags package is correctly installed on the computer that you are using.

Type `source("JessicaTheDAGwithData.R")`. If you get tables and graphics showing that the Monte Carlo answers of rjags approximately match the exact answers then all is well. **The results are best viewed when the graphics window is maximised.** If you do not know how to do this then please let the lecturer know and he will attempt to resolve the problem.

(c) Copy `JessicaTheDAGwithData.R` to a new file named `myJessicaTheDAGwithData.R`

(d) Open `myJessicaTheDAGwithData.R` in a text editor such as (if using `Windows`) `WordPad` (avoid `NotePad`) and change `x1Value <- 2` to `x1Value <- 5` and re-run the script. This corresponds to the data being $\overset{\circ}{x}_1 = 5$ rather than $\overset{\circ}{x}_1 = 2$.

(e) Repeat the previous step for $\overset{\circ}{x}_1 = 1$, $\overset{\circ}{x}_1 = 3$ and $\overset{\circ}{x}_1 = 4$.

(f) Next edit `myJessicaTheDAGwithData.R` to change the potential function from

$$\psi_{123}(x_1, x_2, x_3) = \sqrt{\cosh(x_1 + 1.6\,x_2) + 0.4\,x_3}$$

to

$$\psi_{123}(x_1, x_2, x_3) = \log\left(\frac{8\,x_1 + 17\,x_3}{1 + \sqrt{x_2}}\right)$$

(which can be shown to be positive for all $x_1, x_2, x_3 \in \{1, 2, 3, 4, 5\}$). For this step it is important to note that

  i. The ψ_{123} function is specified twice; once in the JAGS model specification, and once just before the exact calculation.

  ii. In the JAGS model specification code the first function argument is called `x1dash` (and not `x1`) to avoid a variable name clash in the code.

  iii. Both R and JAGS use `sqrt` for the square-root function.

(g) Try one or two of your own potential functions. You need to be sure that it is positive over $x_1, x_2, x_3 \in \{1, 2, 3, 4, 5\}$. Underflow and overflow problems may arise if your ψ_{123} takes very small or very large values (e.g. $\psi_{123}(x_1, x_2, x_3) = \exp[\exp\{107\,x_1\,x_2^6\,x_3^{18}\}]$) so such potential functions are best avoided.

7. Consider

Nathaniel the DAG with data
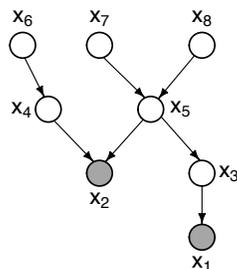
Let $p(x_6)$, $p(x_7)$ and $p(x_8)$ be defined by:

x_6	1	2	3	4	5
$p(x_6)$	0.11	0.35	0.07	0.41	0.06

x_7	1	2	3	4	5
$p(x_7)$	0.21	0.28	0.10	0.18	0.23

x_8	1	2	3	4	5
$p(x_8)$	0.34	0.04	0.25	0.17	0.20

Next, define the potential functions:

$$
\begin{aligned}
\psi_{13}(x_1, x_3) &= \sinh(x_1 + 0.3\,x_3 + 1.8)^2, \\
\psi_{245}(x_2, x_4, x_5) &= (x_2 + 0.7\,x_4 + x_5)^{1/5}, \\
\psi_{35}(x_3, x_5) &= \log\{\cosh(x_3 + x_5 + 2)\}, \\
\psi_{46}(x_4) &= |x_4 - x_6| + 0.134, \\
\psi_{578}(x_5, x_7, x_8) &= \sqrt{(x_5 + 0.7\,x_7)/(0.9 + x_8)}
\end{aligned}
$$

where each

$$ x_i \in \{1, 2, 3, 4, 5\}, \quad i = 1, \ldots, 8. $$

Then set the following the conditional probability functions on Nathaniel the DAG:

$$
\begin{aligned}
p(x_1|x_3) &= \frac{\psi_{13}(x_1, x_3)}{\sum_{x_1'=1}^{5} \psi_{13}(x_1', x_3)}, \\
p(x_2|x_4, x_5) &= \frac{\psi_{245}(x_2, x_4, x_5)}{\sum_{x_2'=1}^{5} \psi_{245}(x_2', x_4, x_5)}, \\
p(x_3|x_5) &= \frac{\psi_{35}(x_3, x_5)}{\sum_{x_3'=1}^{5} \psi_{35}(x_3', x_5)}, \\
p(x_4|x_6) &= \frac{\psi_{46}(x_4, x_6)}{\sum_{x_4'=1}^{5} \psi_{46}(x_4', x_6)}, \\
\text{and} \quad p(x_5|x_7, x_8) &= \frac{\psi_{578}(x_5, x_7, x_8)}{\sum_{x_5'=1}^{5} \psi_{578}(x_5', x_7, x_8)}.
\end{aligned}
\tag{1}
$$

The joint probability function of $x_1, \ldots, x_8$ is then

$$
\begin{aligned}
p(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) &= \prod_{i=1}^{8} p(x_i|\text{parents of } x_i) \\
&= p(x_1|x_3)\, p(x_2|x_4, x_5)\, p(x_3|x_5)\, p(x_4|x_6) \\
&\quad \times p(x_5|x_7, x_8)\, p(x_6)\, p(x_7)\, p(x_8).
\end{aligned}
\tag{2}
$$

The first conditional probability functions corresponding to data

$$ x_1 = \overset{\circ}{x}_1 \quad \text{and} \quad x_2 = \overset{\circ}{x}_2 $$

is

$$
\begin{aligned}
p(x_3|x_1, x_2) &= \frac{p(x_1, x_2, x_3)}{p(x_1, x_2)} \\
&= \frac{\displaystyle\sum_{x_4=1}^{5}\sum_{x_5=1}^{5}\sum_{x_6=1}^{5}\sum_{x_7=1}^{5}\sum_{x_8=1}^{5} p(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)}{\displaystyle\sum_{x_3=1}^{5}\sum_{x_4=1}^{5}\sum_{x_5=1}^{5}\sum_{x_6=1}^{5}\sum_{x_7=1}^{5}\sum_{x_8=1}^{5} p(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)}.
\end{aligned}
$$

So setting $x_1 = \mathring{x}_1$ and $x_2 = \mathring{x}_2$ and substituting into (1) and (2) allows one to get "brute force" expressions for the posterior probability function

$$p(x_3|\mathring{x}_1, \mathring{x}_2).$$

(a) Download the file `NathanielTheDAGwithData.R` and place the file into the same directory/folder where you put the files in Laboratories 1 and 2.

(b) Open an R session and make sure that the current working directory corresponds to the directory/folder in which `NathanielTheDAGwithData.R` resides.

(c) Type `source("NathanielTheDAGwithData.R")`. If you get tables and graphics showing that the Monte Carlo answers of rjags approximately match the exact answers then all is well. **The results are best viewed when the graphics window is maximised.** If you do not know how to do this then please let the lecturer know and he will attempt to resolve the problem.

(d) Copy `NathanielTheDAGwithData.R` to a new file named `myNathanielTheDAGwithData.R`

(e) Open `myNathanielTheDAGwithData.R` in a text editor such as (if using `Windows`) `WordPad` (avoid `NotePad`) and change `x1Value <- 4 ; x2Value <- 2` to `x1Value <- 3 ; x2Value <- 5` and re-run the script. This corresponds to the data being

$\mathring{x}_1$	$\mathring{x}_2$
3	5

rather than

$\mathring{x}_1$	$\mathring{x}_2$
4	2

(f) Repeat the previous step for a few other $(\mathring{x}_1, \mathring{x}_2)$ values.

(g) Try one or two of your own alternative potential functions. You need to be sure that it is positive over $x_1, x_2, x_3 \in \{1, 2, 3, 4, 5\}$.

8. Again consider

Nathaniel the DAG with data
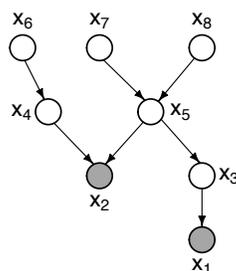
but now suppose that all nodes are continuous random variables with the following distributions:

$$x_1|x_3 \sim t(x_3, 0.92, 8.9), \qquad\qquad x_2|x_4, x_5 \sim N(x_4, (1/x_5^2))$$

$$x_3|x_5 \sim \text{Laplace}(x_5, (1/0.74)), \qquad x_4|x_6 \sim N(0.68, (1/x_6))$$

$$x_5|x_7, x_8 \sim N(x_7, 1/x_8^{1/3}), \qquad\qquad x_6 \sim \text{Beta}(0.12, 0.27)$$

$$x_7 \sim N(3, 1/0.4), \qquad\qquad\qquad x_8 \sim \text{Uniform}(0.1, 0.4)$$

(a) Open an R session and make sure that the current working directory corresponds to the directory/folder in which `summMCMC.r` and `ctsNathanielTheDAGwithData.R` reside.

(b) Type `source("ctsNathanielTheDAGwithData.R")`. This should produce plot that summarises the (Markov chain) Monte Carlo samples and the Monte Carlo-approximate posterior density functions.

Appendix: Individual values of $\wp$ and $f(x_2)$

Varsha Bubathi. $\wp = 0.55$, $\quad f(x_2) = \dfrac{5x_2 + 12}{17x_2 + 19}$.

Tong Zhang. $\wp = 0.53$, $\quad f(x_2) = \dfrac{8x_2 + 11}{9x_2 + 13}$.

Matt Batishall. $\wp = 0.33$, $\quad f(x_2) = \dfrac{6x_2 + 7}{9x_2 + 17}$.

Edwin Lansangan. $\wp = 0.62$, $\quad f(x_2) = \dfrac{7x_2 + 8}{11x_2 + 10}$.

Joschua Braun. $\wp = 0.74$, $\quad f(x_2) = \dfrac{4x_2 + 9}{10x_2 + 13}$.

Virgnia He. $\wp = 0.24$, $\quad f(x_2) = \dfrac{5x_2 + 2}{7x_2 + 12}$.

Rita Do. $\wp = 0.18$, $\quad f(x_2) = \dfrac{5x_2 + 2}{7x_2 + 17}$.

Mark Youssef. $\wp = 0.47$, $\quad f(x_2) = \dfrac{6x_2 + 7}{8x_2 + 14}$.

Greg Hetherington. $\wp = 0.71$, $\quad f(x_2) = \dfrac{4x_2 + 7}{7x_2 + 16}$.

Zorro McCavendish. $\wp = 0.32$, $\quad f(x_2) = \dfrac{4x_2 + 13}{8x_2 + 15}$.