37458 Advanced Bayesian Methods

Some Basics of Distribution Theory

Example 1

$f_{X,Y}(x,y) = x + y, \quad 0 < x < 1, 0 < y < 1.$

What is $f_{Y|X}(y|x)$?

Conditional Distributions

Let X and Y be two random variables.

1. $f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$.

2. $f_{X,Y}(x,y) = f_{Y|X}(y|x) f_X(x)$.

3. $f_Y(y) = \begin{cases} \sum_x f_{X,Y}(x,y) & x \text{ discrete} \\ \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx & x \text{ continuous} \end{cases}$

Answer

$$\begin{aligned} f_X(x) &= \int_0^1 (x+y)\,dy \\ &= \left[xy + \frac{y^2}{2}\right]_0^1 \\ &= x + \tfrac{1}{2}, \; 0 < x < 1. \end{aligned}$$

$$\begin{aligned} \Longrightarrow f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\ &= \frac{x+y}{x+\frac{1}{2}}, \qquad 0 < x < 1,\ 0 < y < 1. \end{aligned}$$

Example 2

$f_{Y|X}(y|x) = \frac{1}{x}, \quad 0 < y < x.$

$f_X(x) = \frac{1}{5}, \quad 0 < x < 5.$

What is $f_Y(y)$?

Answer:

$$
\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx \\
&= \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_X(x)\,dx \\
&= \int_y^5 \frac{1}{x} \cdot \frac{1}{5}\,dx \\
&= \frac{1}{5}\Big[\ln(x)\Big]_y^5 \\
&= \frac{\ln(5) - \ln(y)}{5}, \quad 0 < y < 5
\end{aligned}
$$

Class Exercise

$$
f_{X,Y}(x,y) = \frac{6(x^2 + 3y)}{11},
$$
$$
0 < x < 1, 0 < y < 1.
$$

Determine $f_{Y|X}(y|x)$.

Answer:

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy \\
&= \int_0^1 \frac{6x^2+18y}{11}\,dy \\
&= \left[\frac{6x^2y+9y^2}{11}\right]_0^1 dy \\
&= \frac{6x^2+9}{11}, \quad 0<x<1.
\end{aligned}$$

i.e. $f_X(x) = \dfrac{6x^2+9}{11}, \quad 0<x<1.$

Then

$$\begin{aligned}
f_{Y|X}(y|x) &= \frac{6(x^2+3y)}{11} \Big/ \frac{6x^2+9}{11} \\
&= \frac{2(x^2+3y)}{2x^2+3}
\end{aligned}$$

for $0<x<1, \quad 0<y<1.$

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