ON THE JUSTIFICATION FOR THE VARIABILITY BARS IN AUTOCORRELATION FUNCTION (ACF) PLOTS

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Let $X_1, X_2, \ldots, X_T$ be a stationary time series.

The correlation coefficient at lag k, $k = 0, 1, 2, \ldots, T-1$, is:

$$\rho_k \equiv \frac{\mathrm{Cov}(X_j, X_{j-k})}{\sqrt{\mathrm{Var}(X_j)\mathrm{Var}(X_{j-k})}}$$

for all indices j such that $j, j-k \in \{1, \ldots, T\}$.

Note that each ρ_k has a natural method of moments estimator, $\hat{\rho}_k$ and the ACF plots $\hat{\rho}_k$ against k, $k = 0, 1, \ldots$

(also, trivially, $\rho_0 = \hat{\rho}_0 = 1$)

For any $k=1,2,3,\ldots$ consider the hypotheses:

$$H_0: \rho_k = 0$$

versus $H_1: \rho_k \neq 0.$

Then, under H_0, and some moment conditions on the X_t, $1 \le t \le T$,

$\sqrt{T}\,(\hat{\rho}_k - \rho_k)$ converges in distribution to $N(0,1)$ as $T \to \infty$.

[e.g. Section 7.2 of Time Series: Theory and Methods by P.J. Brockwell & R.A. Davis, Springer 1991.]

Standard arguments then lead to the following approximate 5% level of significance rule:

$$\text{Reject } H_0 \text{ if } |\hat{\rho}_k| > \frac{z_{0.025}}{\sqrt{T}}$$

where $P(Z > z_{0.025}) = 0.025$, $Z \sim N(0,1)$

Note that $z_{0.025} \simeq 1.96 \simeq 2$. Most statistical packages use

$\boxed{\pm \frac{2}{\sqrt{T}} \text{ for the ACF variability bar}}$

based on the above.

EXAMPLE: The object named "sunspots" in the R computing environment is a time series with $T = 2820$ observations.

The R command:

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acf(as.vector(sunspots))
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leads to a plot that looks like: