Supplement for:

The Grouped Horseshoe Distribution and Its Statistical Properties

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S.1 Derivation of Result 1

From (1) we have

$$\mathfrak{p}_{\text{HS},d}(\boldsymbol{x})=\int_0^\infty (2\pi\lambda^2)^{-d/2}\exp\left(-\frac{\|\boldsymbol{x}\|^2}{2\lambda^2}\right)\frac{2}{\pi(1+\lambda^2)}\,d\lambda.$$

The change of variable $t=1/\lambda^2$ then leads to

$$\begin{aligned}\mathfrak{p}_{\text{HS},d}(\boldsymbol{x})&=(2\pi)^{-d/2}\pi^{-1}\int_0^\infty\frac{t^{(d+1)/2-1}\exp(-t\|\boldsymbol{x}\|^2/2)}{1+t}\,dt\\&=(2\pi)^{-d/2}\pi^{-1}\Gamma\big(\tfrac{1}{2}(d+1)\big)\big(\|\boldsymbol{x}\|^2/2\big)^{(1-d)/2}\exp\big(\|\boldsymbol{x}\|^2/2\big)\\&\quad\times\frac{\big(\|\boldsymbol{x}\|^2/2\big)^{(d+1)/2-1}}{\Gamma\big(\tfrac{1}{2}(d+1)\big)}\exp\big(-\|\boldsymbol{x}\|^2/2\big)\int_0^\infty\frac{t^{(d+1)/2-1}\exp(-t\|\boldsymbol{x}\|^2/2)}{1+t}\,dt\\&=\frac{\Gamma\big(\tfrac{1}{2}(d+1)\big)}{\sqrt{2\pi^{d+2}}}\exp\big(\|\boldsymbol{x}\|^2/2\big)E_{(d+1)/2}\big(\|\boldsymbol{x}\|^2/2\big)\Big/\|\boldsymbol{x}\|^{d-1}\end{aligned}$$

with the last step following from 8.19.4 of Olver (2023).

S.2 Derivation of Result 2

We break up the derivation into the cases:

$$d=1\quad\text{and}\quad d\geq 2.$$

<u>*The $d=1$ Case*</u>

As stated in Theorem 1 of Carvalho *et al.* (2010),

$$\mathfrak{p}_{\text{HS},1}(x)>\frac{K_1}{2}\log\left(1+\frac{4}{x^2}\right)\quad\text{where}\quad K_1\equiv\frac{1}{\sqrt{2\pi^3}}.$$

Then

$$\lim_{x\to 0}\mathfrak{p}_{\text{HS},1}(x)>\frac{K_1}{2}\log\left(1+4\lim_{x\to 0}\left(\frac{1}{x^2}\right)\right)=\infty.$$

Hence

$$\lim_{x\to 0}\mathfrak{p}_{\text{HS},1}(x)=\infty$$

and Result 2 holds for $d=1$.

The $d \geq 2$ Case

From Result 1

$$\mathfrak{p}_{\text{HS},d}(\boldsymbol{x}) = K_d \exp\left(\|\boldsymbol{x}\|^2/2\right) E_{(d+1)/2}\left(\|\boldsymbol{x}\|^2/2\right) \Big/ \|\boldsymbol{x}\|^{d-1} \quad \text{where} \quad K_d \equiv \frac{\Gamma\left(\frac{1}{2}(d+1)\right)}{\sqrt{2\pi^{d+2}}}.$$

Then

$$\lim_{\boldsymbol{x}\to\boldsymbol{0}} \mathfrak{p}_{\text{HS},d}(\boldsymbol{x}) = K_d \left\{\lim_{\boldsymbol{x}\to\boldsymbol{0}} \exp\left(\|\boldsymbol{x}\|^2/2\right)\right\} \left\{\lim_{\boldsymbol{x}\to\boldsymbol{0}} E_{(d+1)/2}\left(\|\boldsymbol{x}\|^2/2\right)\right\} \left\{\lim_{\boldsymbol{x}\to\boldsymbol{0}} \|\boldsymbol{x}\|^{1-d}\right\}.$$

Clearly,

$$\lim_{\boldsymbol{x}\to\boldsymbol{0}} \exp\left(\|\boldsymbol{x}\|^2/2\right) = 1.$$

Also, from 8.19.6 of Olver (2023),

$$E_\nu(0) = \frac{1}{\nu - 1}, \quad \text{for all} \quad \nu > 1,$$

which leads to

$$\lim_{\boldsymbol{x}\to\boldsymbol{0}} E_{(d+1)/2}\left(\|\boldsymbol{x}\|^2/2\right) = \frac{1}{\frac{1}{2}(d+1) - 1} = \frac{2}{d-1} \in (0, 2] \quad \text{for all} \quad d \geq 2.$$

Lastly,

$$\lim_{\boldsymbol{x}\to\boldsymbol{0}} \|\boldsymbol{x}\|^{1-d} = \infty \quad \text{for all} \quad d \geq 2.$$

Hence

$$\lim_{\boldsymbol{x}\to\boldsymbol{0}} \mathfrak{p}_{\text{HS},d}(\boldsymbol{x}) = \infty \quad \text{for all} \quad d \geq 2.$$

S.3 Derivation of Result 3

Result 3 entails various properties of the special function known as the bivariate confluent hypergeometric function. We commence with its definition and some key results. We then show how these results lead to the Result 3 statements.

S.3.1 The Bivariate Confluent Hypergeometric Function

The *bivariate confluent hypergeometric function*

$$\Phi_1(\alpha, \beta, \gamma, x, y) \quad \text{for} \quad \alpha, \beta, \gamma, x, y \in \mathbb{C}$$

is defined via a pair of partial differential equations in Section 9.262 of Gradshteyn & Ryzhik (1994). As stated in Section 9.261 of Gradshteyn & Ryzhik (1994) the following series representation applies when $|x| < 1$:

$$\Phi_1(\alpha, \beta, \gamma, x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m x^m y^n}{(\gamma)_{m+n} m! n!} \quad \text{where} \quad (a)_k \equiv \Gamma(a+k)/\Gamma(a). \tag{S.1}$$

Result 3.385 of Gradshteyn & Ryzhik (1994) states that

$$\int_0^1 x^{\nu-1}(1-x)^{\lambda-1}(1-\beta x)^{-\rho} e^{-\mu x}\, dx = \frac{\Gamma(\nu)\Gamma(\lambda)}{\Gamma(\nu+\lambda)} \Phi_1(\nu, \rho, \nu+\lambda, \beta, -\mu) \tag{S.2}$$

for complex numbers λ, ν, ρ, β and μ ranging over various subsets of the complex plane. If these parameters are constrained to be real then the restrictions reduce to

$$\lambda, \nu > 0 \quad \text{and} \quad \beta, \rho, \mu \in \mathbb{R}.$$

Arguments in Appendix B of Gordy (1998) imply that for $0 \leq x < 1$ and $0 < \alpha < \gamma$ we have the following series representation of Φ_1 in terms of the univariate confluent hypergeometric function ${}_1F_1$:

$$\Phi_1(\alpha,\beta,\gamma,x,y) = \exp(y)\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n x^n}{(\gamma)_n\, n!}\,{}_1F_1(\gamma-\alpha,\gamma+n,-y). \tag{S.3}$$

Note, however, that there in an error in equation (6) of Gordy (1998). It is due to the $(\beta)_m$ of (S.1) being incorrectly replaced by $(\beta)_n$. This error leads to (T1) – (T4) of Gordy (1998) containing an incorrect variant of (S.3) with respect to the Φ_1, ${}_1F_1$ and ${}_2F_1$ functions as defined in Gradshteyn & Ryzhik (1994).

S.3.2 Marginal Density Function Simplification

The marginal density function of $\boldsymbol{y}$ according to model (2) is

$$\mathfrak{p}(\boldsymbol{y}) = \int_0^{\infty}\left\{\int_{\mathbb{R}^d}\mathfrak{p}(\boldsymbol{y}|\boldsymbol{\theta})\mathfrak{p}(\boldsymbol{\theta}|\lambda)d\boldsymbol{\theta}\right\}\mathfrak{p}(\lambda)\,d\lambda \quad \text{where} \quad \mathfrak{p}(\lambda) = \frac{2I(\lambda>0)}{\pi(\lambda^2+1)}.$$

Define

$$\xi \equiv 1 + 1/(\lambda^2\tau^2).$$

Then standard algebraic arguments lead to

$$\begin{aligned}\mathfrak{p}(\boldsymbol{y}|\boldsymbol{\theta})\mathfrak{p}(\boldsymbol{\theta}|\lambda) = (2\pi)^{-d/2}(\lambda\tau)^{-d}\exp[\tfrac{1}{2}\{(1/\xi)-1\}\|\boldsymbol{y}\|^2]|(1/\xi)\boldsymbol{I}_d|^{1/2}\\ \times(2\pi)^{-d/2}|(1/\xi)\boldsymbol{I}_d|^{-1/2}\exp\left[-\tfrac{1}{2}\{\boldsymbol{\theta}-(1/\xi)\boldsymbol{y}\}^T\{(1/\xi)\boldsymbol{I}_d\}^{-1}\{\boldsymbol{\theta}-(1/\xi)\boldsymbol{y}\}\right].\end{aligned}$$

Noting that

$$(2\pi)^{-d/2}|(1/\xi)\boldsymbol{I}_d|^{-1/2}\exp\left[-\tfrac{1}{2}\{\boldsymbol{\theta}-(1/\xi)\boldsymbol{y}\}^T\{(1/\xi)\boldsymbol{I}_d\}^{-1}\{\boldsymbol{\theta}-(1/\xi)\boldsymbol{y}\}\right]$$

is the $N\Big((1/\xi)\boldsymbol{y},(1/\xi)\boldsymbol{I}_d\Big)$ density function in $\boldsymbol{\theta}$ and $|(1/\xi)\boldsymbol{I}_d| = \xi^{-d}$, we then have

$$\begin{aligned}\int_{\mathbb{R}^d}\mathfrak{p}(\boldsymbol{y}|\boldsymbol{\theta})\mathfrak{p}(\boldsymbol{\theta}|\lambda)d\boldsymbol{\theta} &= (2\pi)^{-d/2}(\lambda\tau)^{-d}\exp[\tfrac{1}{2}\{(1/\xi)-1\}\|\boldsymbol{y}\|^2]\xi^{-d/2}\\ &= (2\pi)^{-d/2}\exp\left\{-\frac{(\|\boldsymbol{y}\|^2/2)}{1+\lambda^2\tau^2}\right\}\frac{1}{(1+\lambda^2\tau^2)^{d/2}}.\end{aligned}$$

The marginal density function of $\boldsymbol{y}$ is then

$$\mathfrak{p}(\boldsymbol{y}) = (2^{d-2}\pi^{d+2})^{-1/2}\int_0^{\infty}\exp\left\{-\frac{(\|\boldsymbol{y}\|^2/2)}{1+\lambda^2\tau^2}\right\}\frac{1}{(1+\lambda^2\tau^2)^{d/2}(\lambda^2+1)}\,d\lambda. \tag{S.4}$$

S.3.3 Score Function Simplification

The score function is the following $d\times 1$ derivative vector:

$$\nabla_{\boldsymbol{y}}\{\log\mathfrak{p}(\boldsymbol{y})\}.$$

Next note that

$$
\begin{aligned}
d_{\boldsymbol{y}} \log \mathfrak{p}(\boldsymbol{y}) &= \frac{1}{\mathfrak{p}(\boldsymbol{y})} d_{\boldsymbol{y}}\, \mathfrak{p}(\boldsymbol{y}) \\
&= \frac{\displaystyle\int_0^\infty d_{\boldsymbol{y}} \exp\left\{-\frac{(\|\boldsymbol{y}\|^2/2)}{1+\lambda^2\tau^2}\right\} \frac{1}{(1+\lambda^2\tau^2)^{d/2}(\lambda^2+1)}\, d\lambda}{\displaystyle\int_0^\infty \exp\left\{-\frac{(\|\boldsymbol{y}\|^2/2)}{1+\lambda^2\tau^2}\right\} \frac{1}{(1+\lambda^2\tau^2)^{d/2}(\lambda^2+1)}\, d\lambda} \\
&= -\boldsymbol{y} \left[\frac{\displaystyle\int_0^\infty \exp\left\{-\frac{(\|\boldsymbol{y}\|^2/2)}{1+\lambda^2\tau^2}\right\} \frac{1}{(1+\lambda^2\tau^2)^{(d+2)/2}(\lambda^2+1)}\, d\lambda}{\displaystyle\int_0^\infty \exp\left\{-\frac{(\|\boldsymbol{y}\|^2/2)}{1+\lambda^2\tau^2}\right\} \frac{1}{(1+\lambda^2\tau^2)^{d/2}(\lambda^2+1)}\, d\lambda}\right] d\boldsymbol{y}.
\end{aligned}
$$

Hence

$$
\nabla_{\boldsymbol{y}}\{\log \mathfrak{p}(\boldsymbol{y})\} = -\boldsymbol{y} \left[\frac{\displaystyle\int_0^\infty \exp\left\{-\frac{(\|\boldsymbol{y}\|^2/2)}{1+\lambda^2\tau^2}\right\} \frac{1}{(1+\lambda^2\tau^2)^{(d+2)/2}(\lambda^2+1)}\, d\lambda}{\displaystyle\int_0^\infty \exp\left\{-\frac{(\|\boldsymbol{y}\|^2/2)}{1+\lambda^2\tau^2}\right\} \frac{1}{(1+\lambda^2\tau^2)^{d/2}(\lambda^2+1)}\, d\lambda}\right]. \tag{S.5}
$$

S.3.4 Bivariate Confluent Hypergeometric Function Representations

In this subsection we derive expressions for $\mathfrak{p}(\boldsymbol{y})$ and $\nabla_{\boldsymbol{y}}\{\log \mathfrak{p}(\boldsymbol{y})\}$ in terms of the bivariate confluent hypergeometric function Φ_1 as defined in Section S.3.1.

The integral in (S.4) is

$$
\mathcal{C}\big(\tfrac{1}{2}\|\boldsymbol{y}\|^2, \tau^2\big)
$$

where

$$
\mathcal{C}(a,b) \equiv \int_0^\infty \exp\left(-\frac{a}{1+\lambda^2 b}\right) \frac{1}{(1+\lambda^2 b)^{d/2}(\lambda^2+1)}\, d\lambda. \tag{S.6}
$$

The change of variable

$$
x = \lambda^2 b/(1+\lambda^2 b)
$$

in the (S.6) integral leads to

$$
\mathcal{C}(a,b) = \frac{\exp(-a)}{2\sqrt{b}} \int_0^1 x^{\nu-1}(1-x)^{\lambda-1}(1-\beta x)^{-\rho} e^{-\mu\, x}\, dx
$$

where

$$
\nu = \tfrac{1}{2}, \quad \lambda = \tfrac{1}{2}(d+1), \quad \beta = 1 - b^{-1}, \quad \rho = 1 \quad \text{and} \quad \mu = -a.
$$

Application of (S.2) provides the bivariate confluent hypergeometric form

$$
\mathcal{C}(a,b) = \frac{\exp(-a)\sqrt{\pi}\,\Gamma\big(\frac{1}{2}d + \frac{1}{2}\big)}{d\sqrt{b}\,\Gamma\big(\frac{1}{2}d\big)} \Phi_1\Big(\tfrac{1}{2}, 1, \tfrac{1}{2}(d+2), 1-b^{-1}, a\Big).
$$

Plugging this into (S.4), with $a = \frac{1}{2}\|\boldsymbol{y}\|^2$ and $b = \tau^2$, we obtain

$$
\mathfrak{p}(\boldsymbol{y}) = \big(2^{d-2}\pi^{d+1}\big)^{-1/2} \frac{\exp\big(-\frac{1}{2}\|\boldsymbol{y}\|^2\big)\Gamma\big(\frac{1}{2}d + \frac{1}{2}\big)}{\tau\, d\, \Gamma\big(\frac{1}{2}d\big)}\, \Phi_1\left(\tfrac{1}{2}, 1, \tfrac{1}{2}(d+2), 1-\tau^{-2}, \tfrac{1}{2}\|\boldsymbol{y}\|^2\right). \tag{S.7}
$$

For the $d = 1$ special case, (S.7) reduces to an expression similar, but not identical, to that provided by equation (A1) of Carvalho *et al.* (2010). The main difference is an interchange in the fourth and fifth arguments of the Φ_1 function. This discrepancy is attributable to an error in Gordy (1998), which we described in Section S.3.1.

Next we seek an analogous expression for the score function. It follows from (S.5) that

$$\nabla_{\boldsymbol{y}} \log\{\mathfrak{p}(\boldsymbol{y})\} = -\boldsymbol{y}\frac{\mathcal{D}\big(\tfrac{1}{2}\|\boldsymbol{y}\|^2,\tau^2\big)}{\mathcal{C}\big(\tfrac{1}{2}\|\boldsymbol{y}\|^2,\tau^2\big)}$$

where

$$\mathcal{D}(a,b) \equiv \int_0^\infty \exp\left(-\frac{a}{1+\lambda^2 b}\right)\frac{1}{(1+\lambda^2 b)^{(d/2)+1}(\lambda^2+1)}\,d\lambda.$$

Calculations similar to those given in the previous section lead to

$$\mathcal{D}(a,b) = \frac{\exp(-a)}{2\sqrt{b}}\int_0^1 x^{\nu-1}(1-x)^{\lambda-1}(1-\beta x)^{-\rho}e^{-\mu x}\,dx.$$

where

$$\nu = \tfrac{1}{2},\quad \lambda = \tfrac{1}{2}(d+3),\quad \beta = 1-b^{-1},\quad \rho = 1\quad\text{and}\quad \mu = -a.$$

From (S.2),

$$\mathcal{D}(a,b) = \frac{\exp(-a)}{2\sqrt{b}}\frac{\sqrt{\pi}\Gamma\big(\tfrac{1}{2}d+\tfrac{3}{2}\big)}{\Gamma\big(\tfrac{1}{2}\,d+2\big)}\Phi_1\Big(\tfrac{1}{2},1,\tfrac{1}{2}(d+4),1-b^{-1},a\Big).$$

Next note that

$$\Gamma\big(\tfrac{1}{2}d+\tfrac{3}{2}\big) = \Gamma\big(\tfrac{1}{2}d+\tfrac{1}{2}+1\big) = \tfrac{1}{2}(d+1)\Gamma\big(\tfrac{1}{2}d+\tfrac{1}{2}\big)$$

and

$$\Gamma\big(\tfrac{1}{2}d+2\big) = \Gamma\big(\tfrac{1}{2}d+1+1\big) = (\tfrac{1}{2}d+1)\Gamma\big(\tfrac{1}{2}d+1\big).$$

This leads to

$$\frac{\mathcal{D}(a,b)}{\mathcal{C}(a,b)} = \frac{(d+1)\Phi_1\Big(\tfrac{1}{2},1,\tfrac{1}{2}(d+4),1-b^{-1},a\Big)}{(d+2)\Phi_1\Big(\tfrac{1}{2},1,\tfrac{1}{2}(d+2),1-b^{-1},a\Big)}.$$

Hence

$$\nabla_{\boldsymbol{y}} \log\{\mathfrak{p}(\boldsymbol{y})\} = -\frac{\boldsymbol{y}(d+1)\Phi_1\left(\tfrac{1}{2},1,\tfrac{1}{2}(d+4),1-\tau^{-2},\tfrac{1}{2}\|\boldsymbol{y}\|^2\right)}{(d+2)\Phi_1\Big(\tfrac{1}{2},1,\tfrac{1}{2}(d+2),1-\tau^{-2},\tfrac{1}{2}\|\boldsymbol{y}\|^2\Big)}. \tag{S.8}$$

When $d = 1$ this result matches equation (A2) of Carvalho *et al.* (2010) except for interchanges in the fourth and fifth arguments of the Φ_1 function. An error in Gordy (1998), which is described in Section S.3.1, provides an explanation for this discrepancy.

S.3.5 Large $\|\boldsymbol{y}\|$ Approximation of the Score Function

In keeping with Result 3 being concerned with the limiting behaviour of the score function as $\|\boldsymbol{y}\| \to \infty$, throughout this subsection we assume that $\|\boldsymbol{y}\| \gg 1$. The following cases are treated separately (in order of complexity):

$$\tau = 1,\quad \tau > 1\quad\text{and}\quad 0 < \tau < 1.$$

In each case versions of the following result, from e.g. Section 13.1.5 of Abramowitz & Stegun (1968), concerning the right-tail asymptotic behaviour of the univariate confluent hypergeometric function:

$${}_1F_1(a,b,x) = \begin{cases} \dfrac{\Gamma(b)}{\Gamma(a)}\,e^x x^{a-b}\{1+O(|x|^{-1})\} & \text{for } x>0 \text{ and } |x|\gg 1,\\[2ex] \dfrac{\Gamma(b)}{\Gamma(b-a)}(-x)^{-a}\{1+O(|x|^{-1})\} & \text{for } x<0 \text{ and } |x|\gg 1.\end{cases} \tag{S.9}$$

The $\tau = 1$ Case

If $\tau = 1$ it follows from Section 3.383 of Gradshteyn & Ryzhik (1994) that

$$\nabla_{\boldsymbol{y}} \log\{\mathfrak{p}(\boldsymbol{y})\} = -\frac{\boldsymbol{y}(d+1)\,{}_1F_1\left(\frac{1}{2}, \frac{1}{2}(d+4), \frac{1}{2}\|\boldsymbol{y}\|^2\right)}{(d+2)\,{}_1F_1\left(\frac{1}{2}, \frac{1}{2}(d+2), \frac{1}{2}\|\boldsymbol{y}\|^2\right)}. \tag{S.10}$$

From (S.9),

$${}_1F_1\big(\tfrac{1}{2}, \tfrac{1}{2}(d+4), \tfrac{1}{2}\|\boldsymbol{y}\|^2\big) = \frac{\Gamma\big(\frac{1}{2}(d+4)\big)}{\Gamma(\frac{1}{2})} \exp\left(\tfrac{1}{2}\|\boldsymbol{y}\|^2\right)\big(\tfrac{1}{2}\|\boldsymbol{y}\|^2\big)^{-\frac{1}{2}(d+3)}\big\{1 + O(\|\boldsymbol{y}\|^{-2})\big\}$$

and

$${}_1F_1\big(\tfrac{1}{2}, \tfrac{1}{2}(d+2), \tfrac{1}{2}\|\boldsymbol{y}\|^2\big) = \frac{\Gamma\big(\frac{1}{2}(d+2)\big)}{\Gamma(\frac{1}{2})} \exp\left(\tfrac{1}{2}\|\boldsymbol{y}\|^2\right)\big(\tfrac{1}{2}\|\boldsymbol{y}\|^2\big)^{-\frac{1}{2}(d+1)}\big\{1 + O(\|\boldsymbol{y}\|^{-2})\big\}.$$

Therefore,

$$\begin{aligned}\frac{{}_1F_1\big(\frac{1}{2}, \frac{1}{2}(d+4), \frac{1}{2}\|\boldsymbol{y}\|^2\big)}{{}_1F_1\big(\frac{1}{2}, \frac{1}{2}(d+2), \frac{1}{2}\|\boldsymbol{y}\|^2\big)} &= \frac{2\big\{\Gamma\big(\frac{1}{2}(d+4)\big)/\Gamma\big(\frac{1}{2}(d+2)\big)\big\}\|\boldsymbol{y}\|^{-2}\{1 + O(\|\boldsymbol{y}\|^{-2})\}}{1 + O(\|\boldsymbol{y}\|^{-2})} \\ &= \frac{(d+2)\|\boldsymbol{y}\|^{-2}\{1 + O(\|\boldsymbol{y}\|^{-2})\}}{1 + O(\|\boldsymbol{y}\|^{-2})}.\end{aligned}$$

We then have

$$\nabla_{\boldsymbol{y}} \log\{\mathfrak{p}(\boldsymbol{y})\} = -\frac{(d+1)\boldsymbol{y}\|\boldsymbol{y}\|^{-2}\{1 + O(\|\boldsymbol{y}\|^{-2})\}}{1 + O(\|\boldsymbol{y}\|^{-2})} \quad \text{for} \quad \|\boldsymbol{y}\| \gg 1. \tag{S.11}$$

The $\tau > 1$ Case

If $\tau > 1$ then $0 < 1 - \tau^{-2} < 1$ and we can use (S.3) to obtain

$$\begin{aligned}&\Phi_1\left(\tfrac{1}{2}, 1, \tfrac{1}{2}(d+4), 1-\tau^{-2}, \tfrac{1}{2}\|\boldsymbol{y}\|^2\right) \\ &\qquad = \exp\left(\tfrac{1}{2}\|\boldsymbol{y}\|^2\right)\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n(1-\tau^{-2})^n}{\big(\frac{1}{2}(d+4)\big)_n}\,{}_1F_1\Big(\tfrac{1}{2}(d+3), \tfrac{1}{2}(d+4)+n, -\tfrac{1}{2}\|\boldsymbol{y}\|^2\Big).\end{aligned}$$

Next, from (S.9),

$${}_1F_1\Big(\tfrac{1}{2}(d+3), \tfrac{1}{2}(d+4)+n, -\tfrac{1}{2}\|\boldsymbol{y}\|^2\Big) = \frac{\Gamma\big(\frac{1}{2}(d+4)+n\big)}{\Gamma(n+\frac{1}{2})}\{\tfrac{1}{2}\|\boldsymbol{y}\|^2\}^{-(d+3)/2}\big\{1 + O(\|\boldsymbol{y}\|^{-2})\big\}$$

which leads to

$$\Phi_1\left(\tfrac{1}{2}, 1, \tfrac{1}{2}(d+4), 1-\tau^{-2}, \tfrac{1}{2}\|\boldsymbol{y}\|^2\right) = \frac{\tau^2 2^{(d+3)/2}\Gamma(\frac{1}{2}d+2)}{\sqrt{\pi}} \exp\left(\tfrac{1}{2}\|\boldsymbol{y}\|^2\right)\|\boldsymbol{y}\|^{-(d+3)}\big\{1+O(\|\boldsymbol{y}\|^{-2})\big\}.$$

Similarly,

$$\Phi_1\left(\tfrac{1}{2}, 1, \tfrac{1}{2}(d+2), 1-\tau^{-2}, \tfrac{1}{2}\|\boldsymbol{y}\|^2\right) = \frac{\tau^2 2^{(d+1)/2}\Gamma(\frac{1}{2}d+1)}{\sqrt{\pi}} \exp\left(\tfrac{1}{2}\|\boldsymbol{y}\|^2\right)\|\boldsymbol{y}\|^{-(d+1)}\big\{1+O(\|\boldsymbol{y}\|^{-2})\big\}.$$

Substitution into (S.8) then leads to

$$\nabla_{\boldsymbol{y}} \log\{\mathfrak{p}(\boldsymbol{y})\} = \frac{-(d+1)\boldsymbol{y}\|\boldsymbol{y}\|^{-2}\big\{1 + O\big(\|\boldsymbol{y}\|^{-2}\big)\big\}}{1 + O\big(\|\boldsymbol{y}\|^{-2}\big)} \quad \text{for} \quad \|\boldsymbol{y}\| \gg 1. \tag{S.12}$$

The $0<\tau<1$ Case

Recall from results in Section S.3.4 that

$$\nabla_{\boldsymbol{y}}\log\{\mathfrak{p}(\boldsymbol{y})\} = -\boldsymbol{y}\frac{\mathcal{D}\big(\tfrac{1}{2}\|\boldsymbol{y}\|^2,\tau^2\big)}{\mathcal{C}\big(\tfrac{1}{2}\|\boldsymbol{y}\|^2,\tau^2\big)} \tag{S.13}$$

where

$$\mathcal{C}(a,b)=\frac{\exp(-a)}{2\sqrt{b}}\int_0^1 x^{-\frac{1}{2}}(1-x)^{\frac{1}{2}(d-1)}\{1-(1-b^{-1})x\}^{-1}e^{a\,x}\,dx$$

and

$$\mathcal{D}(a,b)=\frac{\exp(-a)}{2\sqrt{b}}\int_0^1 x^{-\frac{1}{2}}(1-x)^{\frac{1}{2}(d+1)}\{1-(1-b^{-1})x\}^{-1}e^{a\,x}\,dx.$$

Noting that

$$1-(1-b^{-1})x=b^{-1}\{1-(1-b)(1-x)\}$$

we have

$$\mathcal{C}(a,b)=\tfrac{1}{2}\exp(-a)\sqrt{b}\int_0^1 x^{-\frac{1}{2}}(1-x)^{\frac{1}{2}(d-1)}\{1-(1-b)(1-x)\}^{-1}e^{a\,x}\,dx.$$

The change of variables $u=1-x$ leads to

$$\mathcal{C}(a,b)=\tfrac{1}{2}\sqrt{b}\int_0^1 u^{\nu-1}(1-u)^{\lambda-1}(1-\beta u)^{-\rho}e^{-\mu u}\,du$$

where

$$\nu=\tfrac{1}{2}(d+1),\quad \lambda=\tfrac{1}{2},\quad \beta=1-b,\quad \rho=1\quad\text{and}\quad \mu=a.$$

Hence, from (S.2),

$$\mathcal{C}(a,b)=\frac{\sqrt{b\pi}\,\Gamma\big(\tfrac{1}{2}(d+1)\big)}{2\Gamma\big(\tfrac{1}{2}(d+2)\big)}\Phi_1\Big(\tfrac{1}{2}(d+1),1,\tfrac{1}{2}(d+2),1-b,-a\Big).$$

Similar steps lead to

$$\mathcal{D}(a,b)=\frac{\sqrt{b\pi}\,\Gamma\big(\tfrac{1}{2}(d+3)\big)}{2\Gamma\big(\tfrac{1}{2}(d+4)\big)}\Phi_1\Big(\tfrac{1}{2}(d+3),1,\tfrac{1}{2}(d+4),1-b,-a\Big).$$

Substitution of these alternative $\mathcal{C}(a,b)$ and $\mathcal{D}(a,b)$ expressions into (S.13) then gives

$$\nabla_{\boldsymbol{y}}\log\{\mathfrak{p}(\boldsymbol{y})\}=-\boldsymbol{y}\,\frac{(d+1)\Phi_1\Big(\tfrac{1}{2}(d+3),1,\tfrac{1}{2}(d+4),1-\tau^2,-\tfrac{1}{2}\|\boldsymbol{y}\|^2\Big)}{(d+2)\Phi_1\Big(\tfrac{1}{2}(d+1),1,\tfrac{1}{2}(d+2),1-\tau^2,-\tfrac{1}{2}\|\boldsymbol{y}\|^2\Big)}.$$

From Appendix B of Gordy (1998), and noting that $0<1-\tau^2<1$,

$$\begin{aligned}&\Phi_1\Big(\tfrac{1}{2}(d+3),1,\tfrac{1}{2}(d+4),1-\tau^2,-\tfrac{1}{2}\|\boldsymbol{y}\|^2\Big)\\&\quad=\exp\big(-\tfrac{1}{2}\|\boldsymbol{y}\|^2\big)\sum_{n=0}^{\infty}\frac{\big(\tfrac{1}{2}(d+3)\big)_n(1-\tau^2)^n}{\big(\tfrac{1}{2}(d+4)\big)_n}\,{}_1F_1\big(\tfrac{1}{2},\tfrac{1}{2}(d+4)+n,\tfrac{1}{2}\|\boldsymbol{y}\|^2\big).\end{aligned} \tag{S.14}$$

Next note that, from Section 13.1.5 of (S.9),

$${}_1F_1\big(\tfrac{1}{2},\tfrac{1}{2}(d+4)+n,\tfrac{1}{2}\|\boldsymbol{y}\|^2\big)=\frac{\Gamma\big(\tfrac{1}{2}(d+4)+n\big)}{\Gamma\big(\tfrac{1}{2}\big)}\exp\big(\tfrac{1}{2}\|\boldsymbol{y}\|^2\big)\big(\tfrac{1}{2}\|\boldsymbol{y}\|^2\big)^{-\frac{1}{2}(d+3)-n}\{1+O(\|\boldsymbol{y}\|^{-2})\}.$$

Substitution into (S.14) then gives

$$\begin{aligned}
&\Phi_1\Big(\tfrac{1}{2}(d+3), 1, \tfrac{1}{2}(d+4), 1-\tau^2, -\tfrac{1}{2}\|\boldsymbol{y}\|^2\Big) \\
&\quad = \frac{2^{(d+3)/2}\Gamma\big(\tfrac{1}{2}(d+4)\big)}{\Gamma\big(\tfrac{1}{2}(d+3)\big)\sqrt{\pi}}\|\boldsymbol{y}\|^{-(d+3)}\big\{1+O(\|\boldsymbol{y}\|^{-2})\big\}\sum_{n=0}^{\infty}\frac{\Gamma\big(\tfrac{1}{2}(d+3)+n\big)\{2(1-\tau^2)\}^n}{\|\boldsymbol{y}\|^{2n}} \\
&\quad = \frac{2^{(d+3)/2}\Gamma\big(\tfrac{1}{2}(d+4)\big)}{\sqrt{\pi}}\|\boldsymbol{y}\|^{-(d+3)}\big\{1+O(\|\boldsymbol{y}\|^{-2})\big\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\Phi_1\Big(\tfrac{1}{2}(d+1), 1, \tfrac{1}{2}(d+2), 1-\tau^2, -\tfrac{1}{2}\|\boldsymbol{y}\|^2\Big) \\
&\qquad = \exp\big(-\tfrac{1}{2}\|\boldsymbol{y}\|^2\big)\sum_{n=0}^{\infty}\frac{\big(\tfrac{1}{2}(d+1)\big)_n(1-\tau^2)^n}{\big(\tfrac{1}{2}(d+2)\big)_n}\,{}_1F_1\big(\tfrac{1}{2}, \tfrac{1}{2}(d+2)+n, \tfrac{1}{2}\|\boldsymbol{y}\|^2\big).
\end{aligned} \tag{S.15}$$

Again, from (S.9),

$${}_1F_1\big(\tfrac{1}{2}, \tfrac{1}{2}(d+2)+n, \tfrac{1}{2}\|\boldsymbol{y}\|^2\big) = \frac{\Gamma\big(\tfrac{1}{2}(d+2)+n\big)}{\Gamma(\tfrac{1}{2})}\exp\big(\tfrac{1}{2}\|\boldsymbol{y}\|^2\big)\big(\tfrac{1}{2}\|\boldsymbol{y}\|^2\big)^{-\frac{1}{2}(d+1)-n}\big\{1+O(\|\boldsymbol{y}\|^{-2})\big\}.$$

Substitution into (S.15) leads to

$$\begin{aligned}
&\Phi_1\Big(\tfrac{1}{2}(d+1), 1, \tfrac{1}{2}(d+2), 1-\tau^2, -\tfrac{1}{2}\|\boldsymbol{y}\|^2\Big) \\
&\quad = \frac{2^{(d+1)/2}\Gamma\big(\tfrac{1}{2}(d+2)\big)}{\Gamma\big(\tfrac{1}{2}(d+1)\big)\sqrt{\pi}}\|\boldsymbol{y}\|^{-(d+1)}\big\{1+O(\|\boldsymbol{y}\|^{-2})\big\}\sum_{n=0}^{\infty}\frac{\Gamma\big(\tfrac{1}{2}(d+1)+n\big)\{2(1-\tau^2)\}^n}{\|\boldsymbol{y}\|^{2n}} \\
&\quad = \frac{2^{(d+1)/2}\Gamma\big(\tfrac{1}{2}(d+2)\big)}{\sqrt{\pi}}\|\boldsymbol{y}\|^{-(d+1)}\big\{1+O(\|\boldsymbol{y}\|^{-2})\big\}.
\end{aligned}$$

We then have

$$\frac{\Phi_1\Big(\tfrac{1}{2}(d+3), 1, \tfrac{1}{2}(d+4), 1-\tau^2, -\tfrac{1}{2}\|\boldsymbol{y}\|^2\Big)}{\Phi_1\Big(\tfrac{1}{2}(d+1), 1, \tfrac{1}{2}(d+2), 1-\tau^2, -\tfrac{1}{2}\|\boldsymbol{y}\|^2\Big)} = \frac{(d+2)\|\boldsymbol{y}\|^{-2}\big\{1+O(\|\boldsymbol{y}\|^{-2})\big\}}{\big\{1+O(\|\boldsymbol{y}\|^{-2})\big\}}.$$

This leads to

$$\nabla_{\boldsymbol{y}}\log\{\mathfrak{p}(\boldsymbol{y})\} = \frac{-(d+1)\boldsymbol{y}\|\boldsymbol{y}\|^{-2}\big\{1+O\big(\|\boldsymbol{y}\|^{-2}\big)\big\}}{1+O\big(\|\boldsymbol{y}\|^{-2}\big)} \quad \text{for} \quad \|\boldsymbol{y}\| \gg 1. \tag{S.16}$$

S.3.6 Tail Limit of the Score Function

Results (S.11), (S.12) and (S.16) imply that, for all $\tau > 0$, the score has the following leading term tail behaviour:

$$\nabla_{\boldsymbol{y}}\log\{\mathfrak{p}(\boldsymbol{y})\} \sim -\frac{(d+1)\boldsymbol{y}}{\|\boldsymbol{y}\|^2} \quad \text{for} \quad \|\boldsymbol{y}\| \gg 1. \tag{S.17}$$

It follows immediately that

$$\lim_{\|\boldsymbol{y}\|\to\infty}\nabla_{\boldsymbol{y}}\log\{\mathfrak{p}(\boldsymbol{y})\} = \boldsymbol{0}.$$

S.3.7 Explicit Expressions for $E(\boldsymbol{\theta}|\boldsymbol{y})$

Arguments similar to those used in Section S.3.4 for the score function lead to

$$E(\boldsymbol{\theta}|\boldsymbol{y})=\frac{\Phi_1\Big(\frac{3}{2},1,\frac{1}{2}(d+4),1-\tau^{-2},\frac{1}{2}\|\boldsymbol{y}\|^2\Big)}{(d+2)\Phi_1\left(\frac{1}{2},1,\frac{1}{2}(d+2),1-\tau^{-2},\frac{1}{2}\|\boldsymbol{y}\|^2\right)}\,\boldsymbol{y}.$$

An alternative expression, that uses an integration by parts step as described in the *Proof of Theorem 2* section of Carvalho *et al.* (2010), is

$$E(\boldsymbol{\theta}|\boldsymbol{y})=\left\{1-\frac{(d+1)\Phi_1\Big(\frac{1}{2},1,\frac{1}{2}(d+4),1-\tau^{-2},\frac{1}{2}\|\boldsymbol{y}\|^2\Big)}{(d+2)\Phi_1\left(\frac{1}{2},1,\frac{1}{2}(d+2),1-\tau^{-2},\frac{1}{2}\|\boldsymbol{y}\|^2\right)}\right\}\boldsymbol{y} \tag{S.18}$$

S.3.8 Bounding of $\big\|\boldsymbol{y}-E(\boldsymbol{\theta}|\boldsymbol{y})\big\|$

It follows from (S.8) and (S.18) that

$$\big\|\boldsymbol{y}-E(\boldsymbol{\theta}|\boldsymbol{y})\big\|=\big\|\nabla_{\boldsymbol{y}}\log\{\mathfrak{p}(\boldsymbol{y})\}\big\|. \tag{S.19}$$

Because of (S.17) we then have

$$\nabla_{\boldsymbol{y}}\log\{\mathfrak{p}(\boldsymbol{y})\}=\mathbf{0}\text{ at }\boldsymbol{y}=\mathbf{0}\quad\text{and}\quad\nabla_{\boldsymbol{y}}\log\{\mathfrak{p}(\boldsymbol{y})\}\sim-\frac{(d+1)\boldsymbol{y}}{\|\boldsymbol{y}\|^2}\quad\text{for}\quad\|\boldsymbol{y}\|\gg 1.$$

Relationship (S.19) then provides

$$\big\|\boldsymbol{y}-E(\boldsymbol{\theta}|\boldsymbol{y})\big\|=0\text{ at }\boldsymbol{y}=\mathbf{0}\quad\text{and}\quad\big\|\boldsymbol{y}-E(\boldsymbol{\theta}|\boldsymbol{y})\big\|\sim\frac{(d+1)}{\|\boldsymbol{y}\|}\quad\text{for}\quad\|\boldsymbol{y}\|\gg 1.$$

This result and the fact that $\big\|\boldsymbol{y}-E(\boldsymbol{\theta}|\boldsymbol{y})\big\|$ is continuous in $\boldsymbol{y}$ implies that $E\big\|\boldsymbol{y}-E(\boldsymbol{\theta}|\boldsymbol{y})\big\|$ is bounded by some $b_\tau<\infty$ that depends only on τ.

S.4 Derivation of Result 4

Consider the Bayesian model

$$\mathfrak{p}(\boldsymbol{y}_1,\ldots,\boldsymbol{y}_n|\boldsymbol{\theta})=\prod_{i=1}^{n}\mathfrak{p}(\boldsymbol{y}_i|\boldsymbol{\theta}),\quad \boldsymbol{\theta}\text{ has prior density function }\mathfrak{p}_{\text{HS},d}(\boldsymbol{\theta}),$$

where, for $1\le i\le n$,

$$\mathfrak{p}(\boldsymbol{y}_i|\boldsymbol{\theta})\equiv(2\pi\sigma^2)^{-d/2}\exp\left\{-\frac{\|\boldsymbol{y}_i-\boldsymbol{\theta}\|^2}{2\sigma^2}\right\}.$$

Suppose that $\boldsymbol{\theta}^0$ is the true value of $\boldsymbol{\theta}$. For each $\boldsymbol{\theta}\in\mathbb{R}^d$, the Kullback-Leibler divergence of $\mathfrak{p}(\boldsymbol{y}_i|\boldsymbol{\theta})$ from $\mathfrak{p}(\boldsymbol{y}_i|\boldsymbol{\theta}^0)$ is

$$\text{KL}\Big(\mathfrak{p}(\boldsymbol{y}_i|\boldsymbol{\theta}^0)\|\mathfrak{p}(\boldsymbol{y}_i|\boldsymbol{\theta})\Big)=\frac{\|\boldsymbol{\theta}-\boldsymbol{\theta}_0\|^2}{2\sigma^2}.$$

For each $\varepsilon>0$ let

$$A_\varepsilon\equiv\Big\{\boldsymbol{\theta}:\text{KL}\Big(\mathfrak{p}(\boldsymbol{y}_i|\boldsymbol{\theta}^0)\|\mathfrak{p}(\boldsymbol{y}_i|\boldsymbol{\theta})\Big)\le\varepsilon\Big\}=\{\|\boldsymbol{\theta}-\boldsymbol{\theta}^0\|^2\le 2\sigma^2\varepsilon\}.$$

Application Proposition 1 of Bhadra *et al.* (2017), which is established in Barron (1987), with $\varepsilon=1/n$ leads to

$$R_n\le\frac{1}{n}-\frac{1}{n}\log\left(\int_{A_{1/n}}\mathfrak{p}_{\text{HS},d}(\boldsymbol{\theta})\,d\boldsymbol{\theta}\right)=\frac{1}{n}-\frac{1}{n}\log\left(\int_{\|\boldsymbol{\theta}-\boldsymbol{\theta}^0\|\le\sigma\sqrt{2/n}}\mathfrak{p}_{\text{HS},d}(\boldsymbol{\theta})\,d\boldsymbol{\theta}\right). \tag{S.20}$$

S.4.1 The $\boldsymbol{\theta}^0 = \mathbf{0}$ Case

For the $\boldsymbol{\theta}^0 = \mathbf{0}$ case (S.20) reduces to

$$R_n \leq \frac{1}{n} - \frac{1}{n}\log\left(\int_{\|\boldsymbol{\theta}\|\leq\sigma\sqrt{2/n}} \mathfrak{p}_{\mathrm{HS},d}(\boldsymbol{\theta})\,d\boldsymbol{\theta}\right). \tag{S.21}$$

To determine the order of magnitude of the right-hand side of (S.21) we consider separately (a) $d = 1$ and (b) $d \geq 2$.

S.4.1.1 The $d = 1$ Case

When $d = 1$ the bound in (S.21) becomes

$$\begin{aligned} R_n &\leq \frac{1}{n} - \frac{1}{n}\log\left(\sqrt{2/\pi^3}\int_0^{\sigma\sqrt{2/n}} \exp(\theta^2/2)\,E_1(\theta^2/2)\,d\theta\right) \\ &= \frac{1}{n} - \frac{1}{n}\log\left(\pi^{-3/2}\int_0^{\sigma^2/n} u^{-1/2}\exp(u)\,E_1(u)\,du\right). \end{aligned} \tag{S.22}$$

From Section 8.214 of Gradshteyn & Ryzhik (1994),

$$E_1(u) = -\gamma - \log(u) - \sum_{k=1}^{\infty}\frac{(-u)^k}{k(k!)} \tag{S.23}$$

where $\gamma \equiv -\mathrm{digamma}(1)$ is Euler's constant. Combining (S.23) with the Taylor series expansion of $\exp(u)$ we obtain

$$\begin{aligned} \int_0^{\sigma^2/n} u^{-1/2}\exp(u)\,E_1(u)\,du &= \int_0^{\sigma^2/n} \big\{ -\gamma u^{-1/2} - u^{-1/2}\log(u) \\ &\qquad + (1-\gamma)u^{1/2} - u^{1/2}\log(u) + \tfrac{1}{4}(3-2\gamma)u^{3/2} + \ldots \big\}\,du \\ &= 2\sigma\log(n)\,n^{-1/2} + O(n^{-1/2}). \end{aligned}$$

It follows that

$$\log\left(\pi^{-3/2}\int_0^{\sigma^2/n} u^{-1/2}\exp(u)\,E_1(u)\,du\right) = -\tfrac{1}{2}\log(n) + \log\{\log(n)\} + O(1).$$

Substitution into (S.22) then leads to

$$R_n \leq \frac{\log(n)}{2n} - \frac{\log\{\log(n)\}}{n} + O\left(\frac{1}{n}\right).$$

S.4.1.2 The $d \geq 2$ Case

In this section we assume that $d \geq 2$. To analyze

$$\mathfrak{I}(d, n, \sigma) \equiv \int_{\|\boldsymbol{\theta}\|\leq\sigma\sqrt{2/n}} \mathfrak{p}_{\mathrm{HS},d}(\boldsymbol{\theta})\,d\boldsymbol{\theta}$$

we switch to hyper-spherical coordinates as follows:

$$\begin{aligned}
\theta_1 &= r\cos(\phi_1),\\
\theta_2 &= r\sin(\phi_1)\cos(\phi_2),\\
\theta_3 &= r\sin(\phi_1)\sin(\phi_2)\cos(\phi_3),\\
&\vdots\\
\theta_{d-1} &= r\sin(\phi_1)\cdots\sin(\phi_{d-2})\cos(\phi_{d-1}),\\
\theta_d &= r\sin(\phi_1)\cdots\sin(\phi_{d-2})\sin(\phi_{d-1})
\end{aligned}$$

where $r \geq 0$, $0 \leq \phi_1, \phi_2, \ldots, \phi_{d-2} \leq \pi$ and $0 \leq \phi_{d-1} < 2\pi$. Then, noting that $\|\boldsymbol{\theta}\| = r$ and the determinant of the Jacobian of the transformation is such that

$$d\boldsymbol{\theta} = r^{d-1}\sin^{d-2}(\phi_1)\sin^{d-3}(\phi_2)\cdots\sin(\phi_{d-2})dr\,d\phi_1\cdots d\phi_{d-1}$$

application of Wallis' Theorem and some additional, but straightforward, algebra leads to

$$\mathfrak{J}(d,n,\sigma) = \frac{\Gamma\big((d+1)/2\big)}{\pi\Gamma(d/2)}\int_0^{\sigma^2/n} u^{-1/2}\exp(u)E_{(d+1)/2}(u)\,du \tag{S.24}$$

The next step involves approximation of the integral in (S.24) using series expansions of $E_{(d+1)/2}(u)$ and then applying results such as

$$\begin{aligned}
&\int_0^{\sigma^2/n} u^{-1/2}\,du = 2\sigma n^{-1/2},\\
&\int_0^{\sigma^2/n} u^{1/2}\log(u)\,du = -\tfrac{2}{3}\sigma^3\log(n)n^{-3/2} + \tfrac{2}{9}\{6\log(\sigma)-2\}\sigma^3 n^{-3/2},\\
&\int_0^{\sigma^2/n} u^{1/2}\,du = \tfrac{2}{3}\sigma^3 n^{-3/2},\\
&\int_0^{\sigma^2/n} u^{3/2}\log(u)\,du = -\tfrac{2}{5}\sigma^5\log(n)n^{-5/2} + \tfrac{2}{25}\{10\log(\sigma)-2\}\sigma^5 n^{-5/2}\\
\text{and } &\int_0^{\sigma^2/n} u^{3/2}\,du = \tfrac{2}{5}\sigma^5 n^{-5/2}.
\end{aligned} \tag{S.25}$$

<u>*The d Odd Case*</u>

If d is odd then $\frac{1}{2}(d+1) \in \mathbb{N}$ and, from from 8.19.8 of Olver (2023),

$$E_{(d+1)/2}(u) = \frac{(-u)^{(d-1)/2}\{\text{digamma}\big((d+1)/2\big) - \log(u)\}}{\{(d-1)/2\}!} - \sum_{k=0,k\neq(d-1)/2}^{\infty}\frac{2(-u)^k}{(2k-d+1)k!}.$$

This expansion, when combined with the Taylor series expansion of $\exp(u)$, leads to

$$\begin{aligned}
&\int_0^{\sigma^2/n} u^{-1/2}\exp(u)\,E_{(d+1)/2}(u)\,du\\
&=\int_0^{\sigma^2/n}(u^{-1/2}+u^{1/2}+\tfrac{1}{2}u^{3/2}+\tfrac{1}{6}u^{5/2}+\ldots)\\
&\quad\times\left\{\frac{(-u)^{(d-1)/2}\{\text{digamma}\big((d+1)/2\big)-\log(u)\}}{\{(d-1)/2\}!}-\sum_{k=0,k\neq(d-1)/2}^{\infty}\frac{2(-u)^k}{(2k-d+1)k!}\right\}du\\
&=\int_0^{\sigma^2/n}\left\{\frac{(-1)^{(d-1)/2}u^{(d-2)/2}\{\text{digamma}\big((d+1)/2\big)-\log(u)\}}{\{(d-1)/2\}!}\right.\\
&\qquad\qquad\left.-\sum_{k=0,k\neq(d-1)/2}^{\infty}\frac{2(-1)^k u^{k-1/2}}{(2k-d+1)k!}\right\}du\\
&\quad+\int_0^{\sigma^2/n}\left\{\frac{(-1)^{(d-1)/2}u^{d/2}\{\text{digamma}\big((d+1)/2\big)-\log(u)\}}{\{(d-1)/2\}!}\right.\\
&\qquad\qquad\left.-\sum_{k=0,k\neq(d-1)/2}^{\infty}\frac{2(-1)^k u^{k+1/2}}{(2k-d+1)k!}\right\}du+\ldots.
\end{aligned}$$

Application of (S.25) to the early terms in these series of integrals reveals that the leading term of the integral in (S.24) is

$$-\frac{2}{(0-d+1)0!}\int_0^{\sigma^2/n}u^{-1/2}\,du=\left(\frac{4\sigma}{d-1}\right)n^{-1/2}.$$

The second term is

$$O\{n^{-3/2}\log(n)\}\quad\text{if } d=3 \text{ and}\quad O(n^{-3/2})\quad\text{if } d=5,7,9,\ldots.$$

Therefore

$$\int_0^{\sigma^2/n}u^{-1/2}\exp(u)\,E_{(d+1)/2}(u)\,du=\left(\frac{4\sigma}{d-1}\right)n^{-1/2}+O\Big(n^{-3/2}\log(n)^{I(d=3)}\Big)\tag{S.26}$$

for all odd integers d exceeding 1.

<u>*The d Even Case*</u>

If d is even then $\frac{1}{2}(d+1)\notin\mathbb{N}$ and, from 8.19.11 of Olver (2023),

$$\Gamma\big((1-d)/2\big)^{-1}\exp(u)E_{(d+1)/2}(u)=u^{(d-1)/2}\exp(u)-\sum_{k=0}^{\infty}\frac{u^k}{\Gamma\big(k+(3-d)/2\big)}.$$

Hence

$$\begin{aligned}
\Gamma\big((1-d)/2\big)^{-1}\int_0^{\sigma^2/n}u^{-1/2}\exp(u)\,E_{(d+1)/2}(u)\,du&=\int_0^{\sigma^2/n}u^{(d-2)/2}\big(1+u+\tfrac{1}{2}u^2+\ldots\big)\,du\\
&\quad-2\sum_{k=0}^{\infty}\frac{\sigma^{2k+1}n^{-(2k+1)/2}}{(2k+1)\Gamma\big(k+(3-d)/2\big)}\\
&=\frac{-2\sigma n^{-1/2}}{\Gamma\big((3-d)/2\big)}+O\big(n^{-\min(d,3)/2}\big)
\end{aligned}$$

and we have

$$
\begin{aligned}
\int_0^{\sigma^2/n} u^{-1/2}\exp(u)\,E_{(d+1)/2}(u)\,du &= -\frac{2\sigma\Gamma\big((1-d)/2\big)n^{-1/2}}{\Gamma\big(1+(1-d)/2\big)} + O\big(n^{-\min(d,3)/2}\big)\\
&= \left(\frac{4\sigma}{d-1}\right)n^{-1/2} + O\big(n^{-\min(d,3)/2}\big)
\end{aligned}
\tag{S.27}
$$

for all even positive integers d.

From (S.21), (S.26) and (S.27) we have

$$
R_n \le \frac{1}{n} - \frac{1}{n}\log\left(\frac{4\Gamma\big((d+1)/2\big)\sigma n^{-1/2}}{\pi\Gamma(d/2)(d-1)} + O(n^{-1})\right) = \frac{\log(n)}{2n} + O\left(\frac{1}{n}\right)
$$

for all $d \ge 2$.

S.4.2 The $\boldsymbol{\theta}^0 \neq \mathbf{0}$ Case

In the $\boldsymbol{\theta}^0 \neq \mathbf{0}$ case we have the bound

$$
R_n \le \frac{1}{n} - \frac{1}{n}\log\left(\int_{S(\boldsymbol{\theta}^0,\sigma,n)} \mathfrak{p}_{\text{HS},d}(\boldsymbol{\theta})\,d\boldsymbol{\theta}\right). \tag{S.28}
$$

where

$$
S(\boldsymbol{\theta}^0,\sigma,n) \equiv \{\boldsymbol{\theta} : \|\boldsymbol{\theta}-\boldsymbol{\theta}^0\| \le \sigma\sqrt{2/n}\}.
$$

If $C(\boldsymbol{\theta}^0,\sigma,n)$ is the largest hypercube inscribed in $S(\boldsymbol{\theta}^0,\sigma,n)$ then

$$
\int_{S(\boldsymbol{\theta}^0,\sigma,n)} \mathfrak{p}_{\text{HS},d}(\boldsymbol{\theta})\,d\boldsymbol{\theta} \ge \int_{C(\boldsymbol{\theta}^0,\sigma,n)} \mathfrak{p}_{\text{HS},d}(\boldsymbol{\theta})\,d\boldsymbol{\theta}. \tag{S.29}
$$

For sufficiently large n, $\mathfrak{p}_{\text{HS},d}(\boldsymbol{\theta})$ is a very smooth function over $C(\boldsymbol{\theta}^0,\sigma,n)$ and Taylor series arguments can be used to establish that

$$
\int_{C(\boldsymbol{\theta}^0,\sigma,n)} \mathfrak{p}_{\text{HS},d}(\boldsymbol{\theta})\,d\boldsymbol{\theta} = K(d,\boldsymbol{\theta}^0,\sigma)n^{-d/2}\{1+o(1)\} \tag{S.30}
$$

for some positive constant $K(d,\boldsymbol{\theta}^0,\sigma)$. Application of (S.29) and (S.30) to the bound in (S.28) then leads to

$$
R_n \le \frac{d\log(n)}{2n} + O\left(\frac{1}{n}\right) \quad \text{for} \quad \boldsymbol{\theta}^0 \neq \mathbf{0}.
$$

S.5 Derivation of Result 5

We first obtain an explicit expression for the posterior density function of λ. Note that

$$
\mathfrak{p}(\lambda|\boldsymbol{y}) = \frac{\mathfrak{p}(\boldsymbol{y},\lambda)}{\mathfrak{p}(\boldsymbol{y})} = \frac{\int_{\mathbb{R}^d}\mathfrak{p}(\boldsymbol{y},\boldsymbol{\psi},\lambda)d\boldsymbol{\psi}}{\mathfrak{p}(\boldsymbol{y})} = \frac{\mathfrak{p}(\lambda)\int_{\mathbb{R}^d}\mathfrak{p}(\boldsymbol{y}|\boldsymbol{\psi})\mathfrak{p}(\boldsymbol{\psi}|\lambda)d\boldsymbol{\psi}}{\mathfrak{p}(\boldsymbol{y})}.
$$

It is easy to establish that

$$
\mathfrak{p}(\boldsymbol{y}|\boldsymbol{\psi}) = (2\pi\tau_1^2)^{-d/2}\exp\left(-\frac{\|\boldsymbol{y}\|^2}{2\tau_1^2}\right)\exp\left\{(1/\tau_1^2)\begin{bmatrix}\boldsymbol{\psi}\\ \text{vec}(\boldsymbol{\psi}\boldsymbol{\psi}^T)\end{bmatrix}^T\begin{bmatrix}\boldsymbol{y}\\ -\frac{1}{2}\text{vec}(\boldsymbol{I}_d)\end{bmatrix}\right\}
$$

and

$$\mathfrak{p}(\boldsymbol{\psi}|\lambda) = (2\pi\lambda^2\tau_2^2)^{-d/2}\exp\left[\{1/(\lambda^2\tau_2^2)\}\left[\begin{array}{c}\boldsymbol{\psi}\\ \text{vec}(\boldsymbol{\psi}\boldsymbol{\psi}^T)\end{array}\right]^T\left[\begin{array}{c}\mathbf{0}\\ -\frac{1}{2}\text{vec}(\boldsymbol{I}_d)\end{array}\right]\right].$$

Hence

$$\mathfrak{p}(\boldsymbol{y}|\boldsymbol{\psi})\mathfrak{p}(\boldsymbol{\psi}|\lambda) = (2\pi\tau_1^2)^{-d/2}\exp\left(-\frac{\|\boldsymbol{y}\|^2}{2\tau_1^2}\right)(2\pi\lambda^2\tau_2^2)^{-d/2}\exp\left\{\left[\begin{array}{c}\boldsymbol{\psi}\\ \text{vec}(\boldsymbol{\psi}\boldsymbol{\psi}^T)\end{array}\right]^T\left[\begin{array}{c}\boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{array}\right]\right\}$$

where

$$\boldsymbol{\eta}_1 \equiv \boldsymbol{y}/\tau_1^2 \quad \text{and} \quad \boldsymbol{\eta}_2 \equiv -\tfrac{1}{2}\left(\frac{1}{\tau_1^2}+\frac{1}{\lambda^2\tau_2^2}\right)\text{vec}(\boldsymbol{I}_d). \tag{S.31}$$

As a function of $\boldsymbol{\psi}$, $\mathfrak{p}(\boldsymbol{y}|\boldsymbol{\psi})\mathfrak{p}(\boldsymbol{\psi}|\lambda)$ is a Multivariate Normal density function with natural parameters given by (S.31). Therefore, standard results concerning the normalizing factor of the Multivariate Normal family leads to

$$\begin{aligned}\int_{\mathbb{R}^d}\mathfrak{p}(\boldsymbol{y}|\boldsymbol{\psi})\mathfrak{p}(\boldsymbol{\psi}|\lambda)d\boldsymbol{\psi} &= (2\pi\tau_1^2)^{-d/2}\exp\left(-\frac{\|\boldsymbol{y}\|^2}{2\tau_1^2}\right)(\lambda^2\tau_2^2)^{-d/2}\left|\left(\frac{1}{\tau_1^2}+\frac{1}{\lambda^2\tau_2^2}\right)\boldsymbol{I}_d\right|^{-1/2}\\ &\quad\times\exp\left[\frac{1}{2(\tau_1^2)^2}\boldsymbol{y}^T\left\{\left(\frac{1}{\tau_1^2}+\frac{1}{\lambda^2\tau_2^2}\right)\boldsymbol{I}_d\right\}^{-1}\boldsymbol{y}\right]\\ &= (2\pi\tau_1^2)^{-d/2}\left(1+\frac{\lambda^2\tau_2^2}{\tau_1^2}\right)^{-d/2}\exp\left\{-\frac{\|\boldsymbol{y}\|^2}{2(\tau_1^2+\lambda^2\tau_2^2)}\right\}.\end{aligned}$$

We then have the following expression for the posterior density function of λ:

$$\mathfrak{p}(\lambda|\boldsymbol{y}) = \frac{2I(\lambda>0)}{\pi(1+\lambda^2)\mathfrak{p}(\boldsymbol{y})}\left\{2\pi(\tau_1^2+\lambda^2\tau_2^2)\right\}^{-d/2}\exp\left\{-\frac{\|\boldsymbol{y}\|^2}{2(\tau_1^2+\lambda^2\tau_2^2)}\right\}.$$

The posterior density function of $\boldsymbol{\psi}$ is

$$\mathfrak{p}(\boldsymbol{\psi}|\boldsymbol{y}) = \frac{\mathfrak{p}(\boldsymbol{y},\boldsymbol{\psi})}{\mathfrak{p}(\boldsymbol{y})} = \frac{\int_0^\infty\mathfrak{p}(\boldsymbol{y},\boldsymbol{\psi},\lambda)d\lambda}{\mathfrak{p}(\boldsymbol{y})} = \frac{\mathfrak{p}(\boldsymbol{y}|\boldsymbol{\psi})\int_0^\infty\mathfrak{p}(\boldsymbol{\psi}|\lambda)\mathfrak{p}(\lambda)d\lambda}{\mathfrak{p}(\boldsymbol{y})}.$$

Introduce following the notation, defined immediately after equation (4) of Wand & Jones (1993):

$$\phi_{\boldsymbol{\Sigma}}(\boldsymbol{x}) = (2\pi)^{-d/2}|\boldsymbol{\Sigma}|^{-1/2}\exp(-\tfrac{1}{2}\boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{x}).$$

Then

$$\begin{aligned}\int_0^\infty\mathfrak{p}(\boldsymbol{\psi}|\lambda)\mathfrak{p}(\lambda)d\lambda &= \int_0^\infty(2\pi\lambda^2\tau_2^2)^{-d/2}\exp\left(-\frac{\|\boldsymbol{\psi}\|^2}{2\lambda^2\tau_2^2}\right)\frac{2}{\pi(1+\lambda^2)}\,d\lambda\\ &= (2/\pi)\int_0^\infty\phi_{\lambda^2\tau_2^2\boldsymbol{I}_d}(\boldsymbol{\psi})\frac{1}{(1+\lambda^2)}\,d\lambda.\end{aligned}$$

Since

$$\mathfrak{p}(\boldsymbol{y}|\boldsymbol{\psi}) = \phi_{\tau_1^2\boldsymbol{I}_d}(\boldsymbol{y}-\boldsymbol{\psi}) = \phi_{\tau_1^2\boldsymbol{I}_d}(\boldsymbol{\psi}-\boldsymbol{y})$$

we then have

$$\mathfrak{p}(\boldsymbol{y})\mathfrak{p}(\boldsymbol{\psi}|\boldsymbol{y}) = (2/\pi)\int_0^\infty\phi_{\tau_1^2\boldsymbol{I}_d}(\boldsymbol{\psi}-\boldsymbol{y})\phi_{\lambda^2\tau_2^2\boldsymbol{I}_d}(\boldsymbol{\psi}-\mathbf{0})\frac{1}{(1+\lambda^2)}\,d\lambda. \tag{S.32}$$

From (A.1) of Wand & Jones (1993),

$$\begin{aligned}\phi_{\tau_1^2 \boldsymbol{I}_d}(\boldsymbol{\psi}-\boldsymbol{y})\phi_{\lambda^2\tau_2^2\boldsymbol{I}_d}(\boldsymbol{\psi}-\boldsymbol{0}) &= \phi_{(\tau_1^2+\lambda^2\tau_2^2)\boldsymbol{I}_d}(\boldsymbol{y})\phi_{\{\lambda^2\tau_1^2\tau_2^2/(\tau_1^2+\lambda^2\tau_2^2)\}\boldsymbol{I}_d}\left(\boldsymbol{\psi}-\frac{\lambda^2\tau_2^2\boldsymbol{y}}{\tau_1^2+\lambda^2\tau_2^2}\right)\\ &= \{2\pi(\tau_1^2+\lambda^2\tau_2^2)\}^{-d/2}\exp\left\{-\frac{\|\boldsymbol{y}\|^2}{2(\tau_1^2+\lambda^2\tau_2^2)}\right\}\\ &\quad\times\phi_{\{\lambda^2\tau_1^2\tau_2^2/(\tau_1^2+\lambda^2\tau_2^2)\}\boldsymbol{I}_d}\left(\boldsymbol{\psi}-\frac{\lambda^2\tau_2^2\,\boldsymbol{y}}{\tau_1^2+\lambda^2\tau_2^2}\right).\end{aligned}$$

Therefore, the posterior density function of $\boldsymbol{\psi}$ has the following expression in terms of the posterior density function of λ:

$$\mathfrak{p}(\boldsymbol{\psi}|\boldsymbol{y}) = \int_0^\infty \mathfrak{p}(\lambda|\boldsymbol{y})\phi_{\{\lambda^2\tau_1^2\tau_2^2/(\tau_1^2+\lambda^2\tau_2^2)\}\boldsymbol{I}_d}\left(\boldsymbol{\psi}-\frac{\lambda^2\tau_2^2\,\boldsymbol{y}}{\tau_1^2+\lambda^2\tau_2^2}\right)d\lambda.$$

Hence, with an interchange in the order of integration,

$$\begin{aligned}E(\boldsymbol{\psi}|\boldsymbol{y}) &= \int_0^\infty \mathfrak{p}(\lambda|\boldsymbol{y})\left\{\int_{\mathbb{R}^d}\boldsymbol{\psi}\phi_{\{\lambda^2\tau_1^2\tau_2^2/(\tau_1^2+\lambda^2\tau_2^2)\}\boldsymbol{I}_d}\left(\boldsymbol{\psi}-\frac{\lambda^2\tau_2^2\,\boldsymbol{y}}{\tau_1^2+\lambda^2\tau_2^2}\right)d\boldsymbol{\psi}\right\}d\lambda\\ &= \int_0^\infty \mathfrak{p}(\lambda|\boldsymbol{y})\left(\frac{\lambda^2\tau_2^2}{\tau_1^2+\lambda^2\tau_2^2}\right)d\lambda\,\boldsymbol{y} = E\left(\frac{\lambda^2\,\tau_2^2}{\tau_1^2+\lambda^2\tau_2^2}\Big|\boldsymbol{y}\right)\boldsymbol{y}\end{aligned}$$

as required.

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