Web-Supplement for:

Solutions to Sparse Multilevel Matrix Problems

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S.1 Proof of Theorem 1

In the case of $m = 2$ the two-level sparse matrix linear system problem is

$$\left[\begin{array}{c|c|c} \boldsymbol{A}_{11} & \boldsymbol{A}_{12,1} & \boldsymbol{A}_{12,2} \\ \hline \boldsymbol{A}_{12,1}^T & \boldsymbol{A}_{22,1} & \boldsymbol{O} \\ \hline \boldsymbol{A}_{12,2}^T & \boldsymbol{O} & \boldsymbol{A}_{22,2} \end{array}\right] \left[\begin{array}{c} \boldsymbol{x}_1 \\ \hline \boldsymbol{x}_{2,1} \\ \hline \boldsymbol{x}_{2,2} \end{array}\right] = \left[\begin{array}{c} \boldsymbol{a}_1 \\ \hline \boldsymbol{a}_{2,1} \\ \hline \boldsymbol{a}_{2,2} \end{array}\right].$$

which immediately leads to

$$\boldsymbol{A}_{11}\boldsymbol{x}_1 + \boldsymbol{A}_{12,1}\boldsymbol{x}_{2,1} + \boldsymbol{A}_{12,2}\,\boldsymbol{x}_{2,2} = \boldsymbol{a}_1$$

and

$$\boldsymbol{A}_{12,i}^T\boldsymbol{x}_1 + \boldsymbol{A}_{22,i}\boldsymbol{x}_{2,i} = \boldsymbol{a}_{2,i}, \quad 1 \le i \le 2.$$

It is clear that the same pattern applies for general m, and we have

$$\boldsymbol{A}_{11}\boldsymbol{x}_1 + \sum_{i=1}^{m}\boldsymbol{A}_{12,i}\boldsymbol{x}_{2,i} = \boldsymbol{a}_1 \tag{S.1}$$

and

$$\boldsymbol{A}_{12,i}^T\boldsymbol{x}_1 + \boldsymbol{A}_{22,i}\boldsymbol{x}_{2,i} = \boldsymbol{a}_{2,i}, \quad 1 \le i \le m. \tag{S.2}$$

Conditions (S.2) immediately imply that

$$\boldsymbol{x}_{2,i} = \boldsymbol{A}_{22,i}^{-1}(\boldsymbol{a}_{2,i} - \boldsymbol{A}_{12,i}^T\boldsymbol{x}_1), \quad 1 \le i \le m. \tag{S.3}$$

Substitution of (S.3) into (S.1) then leads to the solution for $\boldsymbol{x}_1$ as stated in Theorem 1.

For the matrix inverse derivation, we again start with the $m = 2$ case and note that

$$\left[\begin{array}{c|c|c} \boldsymbol{A}_{11} & \boldsymbol{A}_{12,1} & \boldsymbol{A}_{12,2} \\ \hline \boldsymbol{A}_{12,1}^T & \boldsymbol{A}_{22,1} & \boldsymbol{O} \\ \hline \boldsymbol{A}_{12,2}^T & \boldsymbol{O} & \boldsymbol{A}_{22,2} \end{array}\right] \left[\begin{array}{c|c|c} \boldsymbol{A}^{11} & \boldsymbol{A}^{12,1} & \boldsymbol{A}^{12,2} \\ \hline \boldsymbol{A}^{12,1T} & \boldsymbol{A}^{22,1} & \times \\ \hline \boldsymbol{A}^{12,2T} & \times & \boldsymbol{A}^{22,2} \end{array}\right] = \left[\begin{array}{c|c|c} \boldsymbol{I} & \boldsymbol{O} & \boldsymbol{O} \\ \hline \boldsymbol{O} & \boldsymbol{I} & \boldsymbol{O} \\ \hline \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{I} \end{array}\right].$$

Observing the pattern from the $m = 2$ case and then extending to general m we obtain the system of equations:

$$\boldsymbol{A}_{11}\boldsymbol{A}^{11} + \sum_{i=1}^{m}\boldsymbol{A}_{12,i}\boldsymbol{A}^{12,iT} = \boldsymbol{I} \tag{S.4}$$

$$\boldsymbol{A}_{12,i}^T\boldsymbol{A}^{12,i} + \boldsymbol{A}_{22,i}\boldsymbol{A}^{22,i} = \boldsymbol{I}, \quad 1 \le i \le m \tag{S.5}$$

$$\boldsymbol{A}_{12,i}^T\boldsymbol{A}^{11} + \boldsymbol{A}_{22,i}\boldsymbol{A}^{12,iT} = \boldsymbol{O}, \quad 1 \le i \le m. \tag{S.6}$$

From (S.6)

$$\boldsymbol{A}^{12,iT} = -\boldsymbol{A}_{22,i}^{-1}\boldsymbol{A}_{12,i}^T\boldsymbol{A}^{11}, \quad 1 \le i \le m. \tag{S.7}$$

Substitution of (S.7) into (S.4) gives

$$\boldsymbol{A}_{11}\boldsymbol{A}^{11} - \sum_{i=1}^m \boldsymbol{A}_{12,i}\boldsymbol{A}_{22,i}^{-1}\boldsymbol{A}_{12,i}^T\boldsymbol{A}^{11} = \boldsymbol{I}$$

which implies that

$$\boldsymbol{A}^{11} = \left(\boldsymbol{A}_{11} - \sum_{i=1}^m \boldsymbol{A}_{12,i}\boldsymbol{A}_{22,i}^{-1}\boldsymbol{A}_{12,i}^T\right)^{-1}.$$

Substitution of (S.7) into (S.5) gives

$$-\boldsymbol{A}_{12,i}^T\boldsymbol{A}^{11}\boldsymbol{A}_{12,i}\boldsymbol{A}_{22,i}^{-1} + \boldsymbol{A}_{22,i}\boldsymbol{A}^{22,i} = \boldsymbol{I}, \quad 1 \le i \le m$$

implying that

$$\boldsymbol{A}^{22,i} = \boldsymbol{A}_{22,i}^{-1}(\boldsymbol{I} + \boldsymbol{A}_{12,i}^T\boldsymbol{A}^{11}\boldsymbol{A}_{12,i}\boldsymbol{A}_{22,i}^{-1}) = \boldsymbol{A}_{22,i}^{-1}(\boldsymbol{I} - \boldsymbol{A}_{12,i}^T\boldsymbol{A}^{12,i}), \quad 1 \le i \le m.$$

For the $|\boldsymbol{A}|$ result we first proof:

Lemma 1. *Let $\boldsymbol{M}$ be a symmetric invertible matrix with sub-block partioning according to the notation*

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{M}_{11} & \boldsymbol{M}_{12} \\ \boldsymbol{M}_{12}^T & \boldsymbol{M}_{22} \end{bmatrix} \quad \textit{and} \quad \boldsymbol{M}^{-1} = \begin{bmatrix} \boldsymbol{M}^{11} & \boldsymbol{M}^{12} \\ \boldsymbol{M}^{12T} & \boldsymbol{M}^{22} \end{bmatrix}.$$

Then

$$|\boldsymbol{M}| = \left|(\boldsymbol{M}^{11})^{-1}\right| \left|\boldsymbol{M}^{22}\right|.$$

Proof of Lemma 1. Lemma 1 is a direct consequence of Theorem 13.3.8 of Harville (2008) concerning the determinant of a matrix with 2×2 sub-block partitioning.

From Lemma 1 we have

$$|\boldsymbol{A}| = \left|(\boldsymbol{A}^{11})^{-1}\right| \left|\underset{1\le i\le m}{\text{blockdiag}}(\boldsymbol{A}_{22,i})\right| = \left|(\boldsymbol{A}^{11})^{-1}\right| \prod_{i=1}^m |\boldsymbol{A}_{22,i}|.$$

S.2 Proof of Theorem 2

We first note the following simplification:

$$\boldsymbol{B}_i^T\boldsymbol{B}_i = \boldsymbol{B}_i^T\boldsymbol{Q}_i\boldsymbol{Q}_i^T\boldsymbol{B}_i = \boldsymbol{C}_{0i}^T\boldsymbol{C}_{0i} = \boldsymbol{C}_{1i}^T\boldsymbol{C}_{1i} + \boldsymbol{C}_{2i}^T\boldsymbol{C}_{2i}, \quad 1 \le i \le m,$$

where the first equality holds by the orthogonality of $\boldsymbol{Q}_i$ and the second and third equalities hold by Step 1(b) of Theorem 2. A similar sequence of steps can be used to show that

$$\boldsymbol{B}_i^T\dot{\boldsymbol{B}}_i = \boldsymbol{C}_{1i}^T\boldsymbol{R}_i, \quad \dot{\boldsymbol{B}}_i^T\dot{\boldsymbol{B}}_i = \boldsymbol{R}_i^T\boldsymbol{R}_i,$$
$$\boldsymbol{B}_i^T\boldsymbol{b}_i = \boldsymbol{C}_{1i}^T\boldsymbol{c}_{1i} + \boldsymbol{C}_{2i}^T\boldsymbol{c}_{2i} \quad \text{and} \quad \dot{\boldsymbol{B}}_i^T\boldsymbol{b}_i = \boldsymbol{R}_i^T\boldsymbol{c}_{1i}, \; 1 \le i \le m.$$

These simplifications allow us to represent the non-zero components of $\boldsymbol{A}$ and the sub-vectors of $\boldsymbol{a}$ as

$$\boldsymbol{A}_{11} = \sum_{i=1}^m (\boldsymbol{C}_{1i}^T\boldsymbol{C}_{1i} + \boldsymbol{C}_{2i}^T\boldsymbol{C}_{2i}), \quad \boldsymbol{a}_1 = \sum_{i=1}^m (\boldsymbol{C}_{1i}^T\boldsymbol{c}_{1i} + \boldsymbol{C}_{2i}^T\boldsymbol{c}_{2i})$$

and

$$\boldsymbol{A}_{12,i} = \boldsymbol{C}_{1i}^T\boldsymbol{R}_i, \quad \boldsymbol{A}_{22,i} = \boldsymbol{R}_i^T\boldsymbol{R}_i, \quad \boldsymbol{a}_{2,i} = \boldsymbol{R}_i^T\boldsymbol{c}_{1i}, \quad 1 \le i \le m.$$

The derivation of the inverse matrix problem is as follows:

$$
\begin{aligned}
\boldsymbol{A}^{11} &= \left(\boldsymbol{A}_{11} - \sum_{i=1}^{m} \boldsymbol{A}_{12,i}\boldsymbol{A}_{22,i}^{-1}\boldsymbol{A}_{12,i}^{T}\right)^{-1} \\
&= \left(\sum_{i=1}^{m}(\boldsymbol{C}_{1i}^{T}\boldsymbol{C}_{1i} + \boldsymbol{C}_{2i}^{T}\boldsymbol{C}_{2i}) - \sum_{i=1}^{m}\boldsymbol{C}_{1i}^{T}\boldsymbol{R}_i(\boldsymbol{R}_i^{T}\boldsymbol{R}_i)^{-1}\boldsymbol{R}_i^{T}\boldsymbol{C}_{1i}\right)^{-1} \\
&= \left(\sum_{i=1}^{m}\boldsymbol{C}_{2i}^{T}\boldsymbol{C}_{2i}\right)^{-1} \\
&= \left[\left\{\operatorname*{stack}_{1\le i\le m}(\boldsymbol{C}_{2i})\right\}^{T}\left\{\operatorname*{stack}_{1\le i\le m}(\boldsymbol{C}_{2i})\right\}\right]^{-1} \\
&= \left(\begin{bmatrix}\boldsymbol{R}\\ \boldsymbol{O}\end{bmatrix}^{T}\boldsymbol{Q}^{T}\boldsymbol{Q}\begin{bmatrix}\boldsymbol{R}\\ \boldsymbol{O}\end{bmatrix}\right)^{-1} \\
&= \boldsymbol{R}^{-1}\boldsymbol{R}^{-T},
\end{aligned}
$$

where the fifth equality holds by Step 2 of Theorem 2, and we have used the orthogonality of $\boldsymbol{Q}$ for the sixth equality. The other components of $\boldsymbol{A}^{-1}$ are found by simply substituting the above simplifications into Theorem 1.

The derivation of the solution to the linear system is:

$$
\begin{aligned}
\boldsymbol{x}_1 &= \boldsymbol{A}^{11}\left(\boldsymbol{a}_1 - \sum_{i=1}^{m}\boldsymbol{A}_{12,i}\boldsymbol{A}_{22,i}^{-1}\boldsymbol{a}_{2,i}\right) \\
&= \boldsymbol{R}^{-1}\boldsymbol{R}^{-T}\left\{\sum_{i=1}^{m}(\boldsymbol{C}_{1i}^{T}\boldsymbol{c}_{1i} + \boldsymbol{C}_{2i}^{T}\boldsymbol{c}_{2i}) - \sum_{i=1}^{m}\boldsymbol{C}_{1i}^{T}\boldsymbol{R}_i(\boldsymbol{R}_i^{T}\boldsymbol{R}_i)^{-1}\boldsymbol{R}_i^{T}\boldsymbol{c}_{1i}\right\} \\
&= \boldsymbol{R}^{-1}\boldsymbol{R}^{-T}\sum_{i=1}^{m}\boldsymbol{C}_{2i}^{T}\boldsymbol{c}_{2i} \\
&= \boldsymbol{R}^{-1}\boldsymbol{R}^{-T}\left\{\operatorname*{stack}_{1\le i\le m}(\boldsymbol{C}_{2i})\right\}^{T}\left\{\operatorname*{stack}_{1\le i\le m}(\boldsymbol{c}_{2i})\right\} \\
&= \boldsymbol{R}^{-1}\boldsymbol{R}^{-T}\begin{bmatrix}\boldsymbol{R}\\ \boldsymbol{O}\end{bmatrix}^{T}\boldsymbol{Q}^{T}\left\{\operatorname*{stack}_{1\le i\le m}(\boldsymbol{c}_{2i})\right\} \\
&= \boldsymbol{R}^{-1}\boldsymbol{R}^{-T}\boldsymbol{R}^{T}\boldsymbol{c} \\
&= \boldsymbol{R}^{-1}\boldsymbol{c},
\end{aligned}
$$

where we have used Step 2 of Theorem 2 for the fifth and sixth equalities. The other sub-vectors of $\boldsymbol{x}$ are found by simply substituting the above simplifications into Theorem 1.

For the $|\boldsymbol{A}|$ result, the $|\boldsymbol{A}|$ expression from Theorem 1 implies that

$$
|\boldsymbol{A}| = |(\boldsymbol{A}^{11})^{-1}|\prod_{i=1}^{m}|\boldsymbol{A}_{22,i}| = |\boldsymbol{R}^{T}\boldsymbol{R}|\prod_{i=1}^{m}\left|\dot{\boldsymbol{B}}_i^{T}\dot{\boldsymbol{B}}_i\right| = |\boldsymbol{R}^{T}\boldsymbol{R}|\prod_{i=1}^{m}\left|\boldsymbol{R}_i^{T}\boldsymbol{R}_i\right|. \tag{S.8}
$$

Then

$$
|\boldsymbol{R}^{T}\boldsymbol{R}| = |\boldsymbol{R}^{T}||\boldsymbol{R}| = (|\boldsymbol{R}|)^2 = (\text{product of the diagonal entries of } \boldsymbol{R})^2 \tag{S.9}
$$

where we have used the result $|\boldsymbol{M}^{T}| = |\boldsymbol{M}|$ for any square matrix $\boldsymbol{M}$ and the fact that the determinant of an upper-triangular matrix is the product of its diagonal entries (Lemma 13.1.1 of Harville, 2008). Replacement $\boldsymbol{R}$ with $\boldsymbol{R}_i$ in (S.9) and substitution into (S.8) leads to the stated result for $|\boldsymbol{A}|$.

S.3 Proof of Theorem 3

In the case of $m = 2$, $n_1 = 2$ and $n_2 = 3$ the three-level sparse matrix linear system problem is

$$\left[\begin{array}{c|c|c|c|c|c|c|c} \boldsymbol{A}_{11} & \boldsymbol{A}_{12,1} & \boldsymbol{A}_{12,11} & \boldsymbol{A}_{12,12} & \boldsymbol{A}_{12,2} & \boldsymbol{A}_{12,21} & \boldsymbol{A}_{12,22} & \boldsymbol{A}_{12,23} \\ \hline \boldsymbol{A}_{12,1}^T & \boldsymbol{A}_{22,1} & \boldsymbol{A}_{12,1,1} & \boldsymbol{A}_{12,1,2} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} \\ \hline \boldsymbol{A}_{12,11}^T & \boldsymbol{A}_{12,1,1}^T & \boldsymbol{A}_{22,11} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} \\ \hline \boldsymbol{A}_{12,12}^T & \boldsymbol{A}_{12,1,2}^T & \boldsymbol{O} & \boldsymbol{A}_{22,12} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} \\ \hline \boldsymbol{A}_{12,2}^T & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}_{22,2} & \boldsymbol{A}_{12,2,1} & \boldsymbol{A}_{12,2,2} & \boldsymbol{A}_{12,2,3} \\ \hline \boldsymbol{A}_{12,21}^T & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}_{12,2,1}^T & \boldsymbol{A}_{22,21} & \boldsymbol{O} & \boldsymbol{O} \\ \hline \boldsymbol{A}_{12,22}^T & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}_{12,2,2}^T & \boldsymbol{O} & \boldsymbol{A}_{22,22} & \boldsymbol{O} \\ \hline \boldsymbol{A}_{12,23}^T & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}_{12,2,3}^T & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}_{22,23} \end{array}\right] \left[\begin{array}{c} \boldsymbol{x}_1 \\ \hline \boldsymbol{x}_{2,1} \\ \hline \boldsymbol{x}_{2,11} \\ \hline \boldsymbol{x}_{2,12} \\ \hline \boldsymbol{x}_{2,2} \\ \hline \boldsymbol{x}_{2,21} \\ \hline \boldsymbol{x}_{2,22} \\ \hline \boldsymbol{x}_{2,23} \end{array}\right] = \left[\begin{array}{c} \boldsymbol{a}_1 \\ \hline \boldsymbol{a}_{2,1} \\ \hline \boldsymbol{a}_{2,11} \\ \hline \boldsymbol{a}_{2,12} \\ \hline \boldsymbol{a}_{2,2} \\ \hline \boldsymbol{a}_{2,21} \\ \hline \boldsymbol{a}_{2,22} \\ \hline \boldsymbol{a}_{2,23} \end{array}\right].$$

For arbitrary values of m and $\{n_i\}_{1 \le i \le m}$, we immediately obtain the following set of equations:

$$\boldsymbol{A}_{11}\boldsymbol{x}_1 + \sum_{i=1}^{m} \boldsymbol{A}_{12,i}\boldsymbol{x}_{2,i} + \sum_{i=1}^{m}\sum_{j=1}^{n_i} \boldsymbol{A}_{12,ij}\,\boldsymbol{x}_{2,ij} = \boldsymbol{a}_1 \tag{S.10}$$

$$\boldsymbol{A}_{12,i}^T\boldsymbol{x}_1 + \boldsymbol{A}_{22,i}\boldsymbol{x}_{2,i} + \sum_{j=1}^{n_i} \boldsymbol{A}_{12,\,i,\,j}\,\boldsymbol{x}_{2,ij} = \boldsymbol{a}_{2,i}, \quad 1 \le i \le m \tag{S.11}$$

$$\boldsymbol{A}_{12,ij}^T\boldsymbol{x}_1 + \boldsymbol{A}_{12,\,i,\,j}^T\,\boldsymbol{x}_{2,i} + \boldsymbol{A}_{22,ij}\,\boldsymbol{x}_{2,ij} = \boldsymbol{a}_{2,ij}, \quad 1 \le i \le m, \quad 1 \le j \le n_i. \tag{S.12}$$

Conditions (S.12) imply that

$$\boldsymbol{x}_{2,ij} = \boldsymbol{A}_{22,ij}^{-1}\left(\boldsymbol{a}_{2,ij} - \boldsymbol{A}_{12,ij}^T\boldsymbol{x}_1 - \boldsymbol{A}_{12,\,i,\,j}^T\boldsymbol{x}_{2,i}\right), \quad 1 \le i \le m, \quad 1 \le j \le n_i, \tag{S.13}$$

which is the solution for $\boldsymbol{x}_{2,ij}$ as stated in Theorem 3. Substituting this result into conditions (S.11) leads to

$$\boldsymbol{A}_{12,i}^T\boldsymbol{x}_1 + \boldsymbol{A}_{22,i}\boldsymbol{x}_{2,i} + \sum_{j=1}^{n_i} \boldsymbol{A}_{12,\,i,\,j}\boldsymbol{A}_{22,ij}^{-1}\left(\boldsymbol{a}_{2,ij} - \boldsymbol{A}_{12,ij}^T\boldsymbol{x}_1 - \boldsymbol{A}_{12,\,i,\,j}^T\boldsymbol{x}_{2,i}\right) = \boldsymbol{a}_{2,i}, \quad 1 \le i \le m.$$

Solving this equation for $\boldsymbol{x}_{2,i}$ and using the definitions for $\boldsymbol{H}_{12,i}$ and $\boldsymbol{H}_{22,i}$ leads to the solution for $\boldsymbol{x}_{2,i}$ as stated in Theorem 3. Finally, substitution of the results for $\boldsymbol{x}_{2,i}$ and $\boldsymbol{x}_{2,ij}$ into condition (S.10) gives the solution for $\boldsymbol{x}_1$ stated in Theorem 3.

For the matrix inverse, we again illustrate the problem with $m = 2$, $n_1 = 2$ and $n_2 = 3$:

$$
\left[\begin{array}{c|c|c|c|c|c|c|c}
\boldsymbol{A}_{11} & \boldsymbol{A}_{12,1} & \boldsymbol{A}_{12,11} & \boldsymbol{A}_{12,12} & \boldsymbol{A}_{12,2} & \boldsymbol{A}_{12,21} & \boldsymbol{A}_{12,22} & \boldsymbol{A}_{12,23} \\ \hline
\boldsymbol{A}_{12,1}^T & \boldsymbol{A}_{22,1} & \boldsymbol{A}_{12,1,1} & \boldsymbol{A}_{12,1,2} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} \\ \hline
\boldsymbol{A}_{12,11}^T & \boldsymbol{A}_{12,1,1}^T & \boldsymbol{A}_{22,11} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} \\ \hline
\boldsymbol{A}_{12,12}^T & \boldsymbol{A}_{12,1,2}^T & \boldsymbol{O} & \boldsymbol{A}_{22,12} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} \\ \hline
\boldsymbol{A}_{12,2}^T & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}_{22,2} & \boldsymbol{A}_{12,2,1} & \boldsymbol{A}_{12,2,2} & \boldsymbol{A}_{12,2,3} \\ \hline
\boldsymbol{A}_{12,21}^T & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}_{12,2,1}^T & \boldsymbol{A}_{22,21} & \boldsymbol{O} & \boldsymbol{O} \\ \hline
\boldsymbol{A}_{12,22}^T & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}_{12,2,2}^T & \boldsymbol{O} & \boldsymbol{A}_{22,22} & \boldsymbol{O} \\ \hline
\boldsymbol{A}_{12,23}^T & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}_{12,2,3}^T & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}_{22,23}
\end{array}\right]
$$

$$
\times \left[\begin{array}{c|c|c|c|c|c|c|c}
\boldsymbol{A}^{11} & \boldsymbol{A}^{12,1} & \boldsymbol{A}^{12,11} & \boldsymbol{A}^{12,12} & \boldsymbol{A}^{12,2} & \boldsymbol{A}^{12,21} & \boldsymbol{A}^{12,22} & \boldsymbol{A}^{12,23} \\ \hline
\boldsymbol{A}^{12,1T} & \boldsymbol{A}^{22,1} & \boldsymbol{A}^{12,1,1} & \boldsymbol{A}^{12,1,2} & \times & \times & \times & \times \\ \hline
\boldsymbol{A}^{12,11T} & \boldsymbol{A}^{12,1,1T} & \boldsymbol{A}^{22,11} & \times & \times & \times & \times & \times \\ \hline
\boldsymbol{A}^{12,12T} & \boldsymbol{A}^{12,1,2T} & \times & \boldsymbol{A}^{22,12} & \times & \times & \times & \times \\ \hline
\boldsymbol{A}^{12,2T} & \times & \times & \times & \boldsymbol{A}^{22,2} & \boldsymbol{A}^{12,2,1} & \boldsymbol{A}^{12,2,2} & \boldsymbol{A}^{12,2,3} \\ \hline
\boldsymbol{A}^{12,21T} & \times & \times & \times & \boldsymbol{A}^{12,2,1T} & \boldsymbol{A}^{22,21} & \times & \times \\ \hline
\boldsymbol{A}^{12,22T} & \times & \times & \times & \boldsymbol{A}^{12,2,2T} & \times & \boldsymbol{A}^{22,22} & \times \\ \hline
\boldsymbol{A}^{12,23T} & \times & \times & \times & \boldsymbol{A}^{12,2,3T} & \times & \times & \boldsymbol{A}^{22,23}
\end{array}\right]
$$

$$
= \left[\begin{array}{c|c|c|c|c|c|c|c}
\boldsymbol{I} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} \\ \hline
\boldsymbol{O} & \boldsymbol{I} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} \\ \hline
\boldsymbol{O} & \boldsymbol{O} & \boldsymbol{I} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} \\ \hline
\boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{I} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} \\ \hline
\boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{I} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} \\ \hline
\boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{I} & \boldsymbol{O} & \boldsymbol{O} \\ \hline
\boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{I} & \boldsymbol{O} \\ \hline
\boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{I}
\end{array}\right].
$$

Observing the pattern for the $m = 2$, $n_1 = 2$ and $n_2 = 3$ case and extending to general m and

$\{n_i\}_{1\le i\le m}$, we obtain the following system of equations:

$$\boldsymbol{A}_{11}\boldsymbol{A}^{11}+\sum_{i=1}^{m}\boldsymbol{A}_{12,i}\boldsymbol{A}^{12,iT}+\sum_{i=1}^{m}\sum_{j=1}^{n_i}\boldsymbol{A}_{12,ij}\boldsymbol{A}^{12,ijT}=\boldsymbol{I} \tag{S.14}$$

$$\boldsymbol{A}_{12,i}^{T}\boldsymbol{A}^{12,i}+\boldsymbol{A}_{22,i}\boldsymbol{A}^{22,i}+\sum_{j=1}^{n_i}\boldsymbol{A}_{12,\,i,\,j}\boldsymbol{A}^{12,\,i,\,jT}=\boldsymbol{I},\quad 1\le i\le m, \tag{S.15}$$

$$\boldsymbol{A}_{12,ij}^{T}\boldsymbol{A}^{12,ij}+\boldsymbol{A}_{12,\,i,\,j}^{T}\boldsymbol{A}^{12,\,i,\,j}+\boldsymbol{A}_{22,ij}\boldsymbol{A}^{22,ij}=\boldsymbol{I},\quad 1\le i\le m,\quad 1\le j\le n_i, \tag{S.16}$$

$$\boldsymbol{A}_{12,i}^{T}\boldsymbol{A}^{11}+\boldsymbol{A}_{22,i}\boldsymbol{A}^{12,iT}+\sum_{j=1}^{n_i}\boldsymbol{A}_{12,\,i,\,j}\boldsymbol{A}^{12,ijT}=\boldsymbol{O},\quad 1\le i\le m, \tag{S.17}$$

$$\boldsymbol{A}_{12,ij}^{T}\boldsymbol{A}^{11}+\boldsymbol{A}_{12,\,i,\,j}^{T}\boldsymbol{A}^{12,iT}+\boldsymbol{A}_{22,ij}\boldsymbol{A}^{12,ijT}=\boldsymbol{O},\quad 1\le i\le m,\quad 1\le j\le n_i, \tag{S.18}$$

$$\boldsymbol{A}_{12,ij}^{T}\boldsymbol{A}^{12,i}+\boldsymbol{A}_{12,\,i,\,j}^{T}\boldsymbol{A}^{22,i}+\boldsymbol{A}_{22,ij}\boldsymbol{A}^{12,\,i,\,jT}=\boldsymbol{O},\quad 1\le i\le m,\quad 1\le j\le n_i. \tag{S.19}$$

Rearranging conditions (S.18) we obtain

$$\boldsymbol{A}^{12,ij}=-\left(\boldsymbol{A}^{11}\boldsymbol{A}_{12,ij}+\boldsymbol{A}^{12,i}\boldsymbol{A}_{12,\,i,\,j}\right)\boldsymbol{A}_{22,ij}^{-1}\quad 1\le i\le m,\quad 1\le j\le n_i,$$

from which the result for $\boldsymbol{A}^{12,ij}$ stated in Theorem 3 quickly follows. Rearranging conditions (S.17) we obtain

$$\boldsymbol{A}^{12,i}=-\left(\boldsymbol{A}^{11}\boldsymbol{A}_{12,i}+\sum_{j=1}^{n_i}\boldsymbol{A}^{12,ij}\boldsymbol{A}_{12,\,i,\,j}^{T}\right)\boldsymbol{A}_{22,i}^{-1}.$$

Substituting the results for each $\boldsymbol{A}^{12,ij}$ leads to

$$\boldsymbol{A}^{12,i}=-\left\{\boldsymbol{A}^{11}\boldsymbol{A}_{12,i}-\sum_{j=1}^{n_i}\left(\boldsymbol{A}^{11}\boldsymbol{A}_{12,ij}+\boldsymbol{A}^{12,i}\boldsymbol{A}_{12,\,i,\,j}\right)\boldsymbol{A}_{22,ij}^{-1}\boldsymbol{A}_{12,\,i,\,j}^{T}\right\}\boldsymbol{A}_{22,i}^{-1},\ 1\le i\le m,\ 1\le j\le n_i.$$

Solving the above set of equations for each $\boldsymbol{A}^{12,i}$ and using the definitions of $\boldsymbol{H}_{12,i}$ and $\boldsymbol{H}_{22,i}$ we obtain the result in Theorem 3. Substituting the results for each $\boldsymbol{A}^{12,ij}$ and $\boldsymbol{A}^{12,i}$ into (S.14), using the definition of $\boldsymbol{H}_{12,i}$ and solving for $\boldsymbol{A}^{11}$ leads to its stated result in Theorem 3. Rearranging conditions (S.19) we obtain

$$\boldsymbol{A}^{12,\,i,\,j}=-(\boldsymbol{A}^{12,iT}\boldsymbol{A}_{12,ij}+\boldsymbol{A}^{22,i}\boldsymbol{A}_{12,\,i,\,j})\boldsymbol{A}_{22,ij}^{-1},\quad 1\le i\le m,\quad 1\le j\le n_i,$$

from which the result for $\boldsymbol{A}^{12,\,i,\,j}$ stated in Theorem 3 follows quickly. Substitution of these results into the corresponding equations of condition (S.15) leads to

$$\boldsymbol{A}_{12,i}^{T}\boldsymbol{A}^{12,i}+\boldsymbol{A}_{22,i}\boldsymbol{A}^{22,i}-\sum_{j=1}^{n_i}\boldsymbol{A}_{12,\,i,\,j}\boldsymbol{A}_{22,ij}^{-1}(\boldsymbol{A}_{12,ij}^{T}\boldsymbol{A}^{12,i}+\boldsymbol{A}_{12,\,i,\,j}^{T}\boldsymbol{A}^{22,i})=\boldsymbol{I},\quad 1\le i\le m.$$

Solving the above set of equations for each $\boldsymbol{A}^{22,i}$ and using the definitions of $\boldsymbol{H}_{12,i}$ and $\boldsymbol{H}_{22,i}$ in equation (7) of the main article we obtain the result in Theorem 3. Rearrangement of (S.16) leads to

$$\boldsymbol{A}^{22,ij}=\boldsymbol{A}_{22,ij}^{-1}(\boldsymbol{I}-\boldsymbol{A}_{12,ij}^{T}\boldsymbol{A}^{12,ij}-\boldsymbol{A}_{12,\,i,\,j}^{T}\boldsymbol{A}^{12,\,i,\,j}),\quad 1\le i\le m,\quad 1\le j\le n_i.$$

From Lemma 1 in the proof of Theorem 1,

$$|\boldsymbol{A}|=\left|(\boldsymbol{A}^{11})^{-1}\right|\prod_{i=1}^{m}\begin{vmatrix}\boldsymbol{A}_{22,i} & \left\{\underset{1\le j\le n_i}{\text{stack}}\,(\boldsymbol{A}_{12,\,i,\,j}^{T})\right\}^{T}\\ \underset{1\le j\le n_i}{\text{stack}}\,(\boldsymbol{A}_{12,\,i,\,j}^{T}) & \underset{1\le j\le n_i}{\text{blockdiag}}(\boldsymbol{A}_{22,ij})\end{vmatrix}.$$

Application of Lemma 1 and Theorem 1 to the two-level sparse matrices in the last-written expression leads to

$$\left|\begin{array}{cc} \boldsymbol{A}_{22,i} & \left\{\underset{1\le j\le n_i}{\text{stack}}\,(\boldsymbol{A}_{12,\,i,\,j}^T)\right\}^T \\ \underset{1\le j\le n_i}{\text{stack}}\,(\boldsymbol{A}_{12,\,i,\,j}^T) & \underset{1\le j\le n_i}{\text{blockdiag}}(\boldsymbol{A}_{22,ij}) \end{array}\right| = \left|\boldsymbol{A}_{22,i} - \sum_{j=1}^{n_i}\boldsymbol{A}_{12,\,i,\,j}\boldsymbol{A}_{22,ij}^{-1}\boldsymbol{A}_{12,\,i,\,j}^T\right|\prod_{j=1}^{n_i}|\boldsymbol{A}_{22,ij}|$$

and the stated result for $|\boldsymbol{A}|$ follows immediately.

S.4 Proof of Theorem 4

We first note the following re-expression:

$$\boldsymbol{B}_{ij}^T\boldsymbol{B}_{ij} = \boldsymbol{B}_{ij}^T\boldsymbol{Q}_{ij}\boldsymbol{Q}_{ij}^T\boldsymbol{B}_{ij} = \begin{bmatrix}\boldsymbol{D}_{1ij}\\ \boldsymbol{D}_{2ij}\end{bmatrix}^T\begin{bmatrix}\boldsymbol{D}_{1ij}\\ \boldsymbol{D}_{2ij}\end{bmatrix} = \boldsymbol{D}_{1ij}^T\boldsymbol{D}_{1ij} + \boldsymbol{D}_{2ij}^T\boldsymbol{D}_{2ij},\ 1\le i\le m,\quad 1\le j\le n_i,$$

where the first equality holds by orthogonality of $\boldsymbol{Q}_{ij}$ and the second equality holds by Step 1(a)ii of Theorem 4. A similar sequence of steps can be used to show that

$$\begin{aligned}&\dot{\boldsymbol{B}}_{ij}^T\dot{\boldsymbol{B}}_{ij} = \dot{\boldsymbol{D}}_{1ij}^T\dot{\boldsymbol{D}}_{1ij} + \dot{\boldsymbol{D}}_{2ij}^T\dot{\boldsymbol{D}}_{2ij},\quad \boldsymbol{B}_{ij}^T\dot{\boldsymbol{B}}_{ij} = \boldsymbol{D}_{1ij}^T\dot{\boldsymbol{D}}_{1ij} + \boldsymbol{D}_{2ij}^T\dot{\boldsymbol{D}}_{2ij},\\ &\boldsymbol{B}_{ij}^T\boldsymbol{b}_{ij} = \boldsymbol{D}_{1ij}^T\boldsymbol{d}_{1ij} + \boldsymbol{D}_{2ij}^T\boldsymbol{d}_{2ij}\quad\text{and}\quad \dot{\boldsymbol{B}}_{ij}^T\boldsymbol{b}_{ij} = \dot{\boldsymbol{D}}_{1ij}^T\boldsymbol{d}_{1ij} + \dot{\boldsymbol{D}}_{2ij}^T\boldsymbol{d}_{2ij},\quad 1\le i\le m,\quad 1\le j\le n_i.\end{aligned}$$

We also have

$$\ddot{\boldsymbol{B}}_{ij}^T\ddot{\boldsymbol{B}}_{ij} = \ddot{\boldsymbol{B}}_{ij}^T\boldsymbol{Q}_{ij}\boldsymbol{Q}_{ij}^T\ddot{\boldsymbol{B}}_{ij} = \begin{bmatrix}\boldsymbol{R}_{ij}\\ \boldsymbol{0}\end{bmatrix}^T\begin{bmatrix}\boldsymbol{R}_{ij}\\ \boldsymbol{0}\end{bmatrix} = \boldsymbol{R}_{ij}^T\boldsymbol{R}_{ij},$$

where the first equality holds by orthogonality of $\boldsymbol{Q}_{ij}$ and the second equality holds by Step 1(a)i of Theorem 4. A similar sequence of steps can be used to show that

$$\dot{\boldsymbol{B}}_{ij}^T\ddot{\boldsymbol{B}}_{ij} = \dot{\boldsymbol{D}}_{1ij}^T\boldsymbol{R}_{ij},\quad \boldsymbol{B}_{ij}^T\ddot{\boldsymbol{B}}_{ij} = \boldsymbol{D}_{1ij}^T\boldsymbol{R}_{ij}\quad\text{and}\quad \ddot{\boldsymbol{B}}_{ij}^T\boldsymbol{b}_{ij} = \boldsymbol{R}_{ij}^T\boldsymbol{d}_{1ij}.$$

The above simplifications allow us to represent the non-zero components of $\boldsymbol{A}$ and the sub-vectors of $\boldsymbol{a}$ as

$$\begin{aligned}&\boldsymbol{A}_{11} = \sum_{i=1}^m\sum_{j=1}^{n_i}(\boldsymbol{D}_{1ij}^T\boldsymbol{D}_{1ij} + \boldsymbol{D}_{2ij}^T\boldsymbol{D}_{2ij}),\quad \boldsymbol{a}_1 = \sum_{i=1}^m\sum_{j=1}^{n_i}(\boldsymbol{D}_{1ij}^T\boldsymbol{d}_{1ij} + \boldsymbol{D}_{2ij}^T\boldsymbol{d}_{2ij}),\\ &\boldsymbol{A}_{22,i} = \sum_{j=1}^{n_i}(\dot{\boldsymbol{D}}_{1ij}^T\dot{\boldsymbol{D}}_{1ij} + \dot{\boldsymbol{D}}_{2ij}^T\dot{\boldsymbol{D}}_{2ij}),\quad \boldsymbol{A}_{12,i} = \sum_{j=1}^{n_i}(\boldsymbol{D}_{1ij}^T\dot{\boldsymbol{D}}_{1ij} + \boldsymbol{D}_{2ij}^T\dot{\boldsymbol{D}}_{2ij})\\ &\boldsymbol{a}_{2,i} = \sum_{j=1}^{n_i}(\dot{\boldsymbol{D}}_{1ij}^T\boldsymbol{d}_{1ij} + \dot{\boldsymbol{D}}_{2ij}^T\boldsymbol{d}_{2ij}),\quad 1\le i\le m,\end{aligned}$$

and, for $1\le i\le m,\ 1\le j\le n_i$,

$$\boldsymbol{A}_{22,ij} = \boldsymbol{R}_{ij}^T\boldsymbol{R}_{ij},\quad \boldsymbol{A}_{12,\,i,\,j} = \dot{\boldsymbol{D}}_{1ij}^T\boldsymbol{R}_{ij},\quad \boldsymbol{A}_{12,ij} = \boldsymbol{D}_{1ij}^T\boldsymbol{R}_{ij},\quad \boldsymbol{a}_{2,ij} = \boldsymbol{R}_{ij}^T\boldsymbol{d}_{1ij}.$$

Furthermore, each $\boldsymbol{H}_{22,i}$ matrix takes the form

$$\begin{aligned}
\boldsymbol{H}_{22,i} &= \boldsymbol{A}_{22,i} - \sum_{j=1}^{n_i} \boldsymbol{A}_{12,i\ j}\boldsymbol{A}_{22,ij}^{-1}\boldsymbol{A}_{12,\,i,\,j}^{T} \\
&= \sum_{j=1}^{n_i}(\dot{\boldsymbol{D}}_{1ij}^{T}\dot{\boldsymbol{D}}_{1ij} + \dot{\boldsymbol{D}}_{2ij}^{T}\dot{\boldsymbol{D}}_{2ij}) - \sum_{j=1}^{n_i}\dot{\boldsymbol{D}}_{1ij}^{T}\boldsymbol{R}_{ij}(\boldsymbol{R}_{ij}^{T}\boldsymbol{R}_{ij})^{-1}\boldsymbol{R}_{ij}^{T}\dot{\boldsymbol{D}}_{1ij} \\
&= \sum_{j=1}^{n_i}\dot{\boldsymbol{D}}_{2ij}^{T}\dot{\boldsymbol{D}}_{2ij} \\
&= \left\{\underset{1\le j\le n_i}{\text{stack}}\ \dot{\boldsymbol{D}}_{2ij}\right\}^{T}\left\{\underset{1\le j\le n_i}{\text{stack}}\ \dot{\boldsymbol{D}}_{2ij}\right\} \\
&= \left(\boldsymbol{Q}_i\begin{bmatrix}\boldsymbol{R}_i\\ \boldsymbol{0}\end{bmatrix}\right)^{T}\boldsymbol{Q}_i\begin{bmatrix}\boldsymbol{R}_i\\ \boldsymbol{0}\end{bmatrix} \\
&= \begin{bmatrix}\boldsymbol{R}_i\\ \boldsymbol{0}\end{bmatrix}^{T}\begin{bmatrix}\boldsymbol{R}_i\\ \boldsymbol{0}\end{bmatrix} \\
&= \boldsymbol{R}_i^{T}\boldsymbol{R}_i, \quad 1\le i\le m,
\end{aligned}$$

where the fifth equality holds by Step 1(b)ii of Theorem 4. Similarly,

$$\boldsymbol{H}_{12,i} = \boldsymbol{C}_{1i}^{T}\boldsymbol{R}_i \quad \text{and} \quad \boldsymbol{H}_{12,i}\boldsymbol{H}_{22,i}^{-1}\boldsymbol{H}_{12,i}^{T} = \boldsymbol{C}_{1i}^{T}\boldsymbol{C}_{1i}, \quad 1\le i\le m.$$

We are now in a position to derive the solutions to the inverse matrix problem:

$$\begin{aligned}
\boldsymbol{A}^{11} &= \left\{\sum_{i=1}^{m}\sum_{j=1}^{n_i}(\boldsymbol{D}_{1ij}^{T}\boldsymbol{D}_{1ij} + \boldsymbol{D}_{2ij}^{T}\boldsymbol{D}_{2ij}) - \sum_{i=i}^{m}\boldsymbol{C}_{1i}^{T}\boldsymbol{C}_{1i} - \sum_{i=1}^{m}\sum_{j=1}^{n_i}\boldsymbol{D}_{1ij}^{T}\boldsymbol{R}_{ij}(\boldsymbol{R}_{ij}^{T}\boldsymbol{R}_{ij})^{-1}\boldsymbol{R}_{ij}^{T}\boldsymbol{D}_{1ij}\right\}^{-1} \\
&= \left(\sum_{i=1}^{m}\sum_{j=1}^{n_i}\boldsymbol{D}_{2ij}^{T}\boldsymbol{D}_{2ij} - \sum_{i=i}^{m}\boldsymbol{C}_{1i}^{T}\boldsymbol{C}_{1i}\right)^{-1} \\
&= \left(\sum_{i=1}^{m}\left[\left\{\underset{1\le j\le n_i}{\text{stack}}(\boldsymbol{D}_{2ij})\right\}^{T}\left\{\underset{1\le j\le n_i}{\text{stack}}(\boldsymbol{D}_{2ij})\right\} - \boldsymbol{C}_{1i}^{T}\boldsymbol{C}_{1i}\right]\right)^{-1} \\
&= \left(\sum_{i=1}^{m}\left[\left\{\underset{1\le j\le n_i}{\text{stack}}(\boldsymbol{D}_{2ij})\right\}^{T}\boldsymbol{Q}_i\boldsymbol{Q}_i^{T}\left\{\underset{1\le j\le n_i}{\text{stack}}(\boldsymbol{D}_{2ij})\right\} - \boldsymbol{C}_{1i}^{T}\boldsymbol{C}_{1i}\right]\right)^{-1} \\
&= \left(\sum_{i=1}^{m}\left[\begin{bmatrix}\boldsymbol{C}_{1i}\\ \boldsymbol{C}_{2i}\end{bmatrix}^{T}\begin{bmatrix}\boldsymbol{C}_{1i}\\ \boldsymbol{C}_{2i}\end{bmatrix} - \boldsymbol{C}_{1i}^{T}\boldsymbol{C}_{1i}\right]\right)^{-1} \\
&= \left(\sum_{i=1}^{m}\boldsymbol{C}_{2i}^{T}\boldsymbol{C}_{2i}\right)^{-1} \\
&= \left[\left\{\underset{1\le i\le m}{\text{stack}}(\boldsymbol{C}_{2i})\right\}^{T}\left\{\underset{1\le i\le m}{\text{stack}}(\boldsymbol{C}_{2i})\right\}\right]^{-1} \\
&= \left\{\left(\boldsymbol{Q}\begin{bmatrix}\boldsymbol{R}\\ \boldsymbol{0}\end{bmatrix}\right)^{T}\boldsymbol{Q}\begin{bmatrix}\boldsymbol{R}\\ \boldsymbol{0}\end{bmatrix}\right\}^{-1} \\
&= \left(\begin{bmatrix}\boldsymbol{R}\\ \boldsymbol{0}\end{bmatrix}^{T}\begin{bmatrix}\boldsymbol{R}\\ \boldsymbol{0}\end{bmatrix}\right)^{-1} \\
&= \boldsymbol{R}^{-1}\boldsymbol{R}^{-T},
\end{aligned}$$

where the fourth equality holds by orthogonality of each $\boldsymbol{Q}_i$, the fifth equality holds by Step 1(b)ii, the eighth equality holds by Step 2 and the ninth equality holds by orthogonality of $\boldsymbol{Q}$. The other components of $\boldsymbol{A}^{-1}$ are found by simply substituting the above simplifications into Theorem 3 and using a similar sequence of steps.

For the linear system solution we have

$$
\begin{aligned}
\boldsymbol{h}_{2,i} &= \sum_{j=1}^{n_i}(\dot{\boldsymbol{D}}_{1ij}^T\boldsymbol{d}_{1ij} + \dot{\boldsymbol{D}}_{2ij}^T\boldsymbol{d}_{2ij}) - \sum_{j=1}^{n_i}\dot{\boldsymbol{D}}_{1ij}^T\boldsymbol{R}_{ij}(\boldsymbol{R}_{ij}^T\boldsymbol{R}_{ij})^{-1}\boldsymbol{R}_{ij}^T\boldsymbol{d}_{1ij} = \sum_{j=1}^{n_i}\dot{\boldsymbol{D}}_{2ij}^T\boldsymbol{d}_{2ij} \\
&= \left\{\underset{1\le j\le n_i}{\text{stack}}(\dot{\boldsymbol{D}}_{2ij})\right\}^T\left\{\underset{1\le j\le n_i}{\text{stack}}(\boldsymbol{d}_{2ij})\right\} = \begin{bmatrix}\boldsymbol{R}_i\\ \boldsymbol{0}\end{bmatrix}^T\boldsymbol{Q}_i^T\left\{\underset{1\le j\le n_i}{\text{stack}}(\boldsymbol{d}_{2ij})\right\} \\
&= \begin{bmatrix}\boldsymbol{R}_i\\ \boldsymbol{0}\end{bmatrix}^T\begin{bmatrix}\boldsymbol{c}_{1i}\\ \boldsymbol{c}_{2i}\end{bmatrix} = \boldsymbol{R}_i^T\boldsymbol{c}_{1i}, \quad 1\le i\le m,
\end{aligned}
$$

where the fourth and fifth equalities hold by Step 1(b)i and Step 1(b)ii, respectively. Also,

$$
\boldsymbol{A}_{12,ij}\boldsymbol{A}_{22,ij}^{-1}\boldsymbol{a}_{2,ij} = \boldsymbol{D}_{1ij}^T\boldsymbol{R}_{ij}(\boldsymbol{R}_{ij}^T\boldsymbol{R}_{ij})^{-1}\boldsymbol{R}_{ij}^T\boldsymbol{d}_{1ij} = \boldsymbol{D}_{1ij}^T\boldsymbol{d}_{1ij}, \quad 1\le i\le m, \quad 1\le j\le n_i.
$$

The final task is to derive the expressions in Theorem 4 for the components of $\boldsymbol{x}$. For the first block we have:

$$
\begin{aligned}
\boldsymbol{x}_1 &= \boldsymbol{A}^{11}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n_i}(\boldsymbol{D}_{1ij}^T\boldsymbol{d}_{1ij} + \boldsymbol{D}_{2ij}^T\boldsymbol{d}_{2ij}) - \sum_{i=1}^{m}\boldsymbol{C}_{1i}^T\boldsymbol{R}_i^{-T}\boldsymbol{R}_i^T\boldsymbol{c}_{1i} - \sum_{i=1}^{m}\sum_{j=1}^{n_i}\boldsymbol{D}_{1ij}^T\boldsymbol{d}_{1ij}\right\} \\
&= \boldsymbol{A}^{11}\left\{\sum_{i=1}^{m}\left(\sum_{j=1}^{n_i}\boldsymbol{D}_{2ij}^T\boldsymbol{d}_{2ij} - \boldsymbol{C}_{1i}^T\boldsymbol{c}_{1i}\right)\right\} \\
&= \boldsymbol{A}^{11}\left(\sum_{i=1}^{m}\left[\left\{\underset{1\le j\le n_i}{\text{stack}}(\boldsymbol{D}_{2ij})\right\}^T\left\{\underset{1\le j\le n_i}{\text{stack}}(\boldsymbol{d}_{2ij})\right\} - \boldsymbol{C}_{1i}^T\boldsymbol{c}_{1i}\right]\right) \\
&= \boldsymbol{A}^{11}\left(\sum_{i=1}^{m}\left[\left\{\underset{1\le j\le n_i}{\text{stack}}(\boldsymbol{D}_{2ij})\right\}^T\boldsymbol{Q}_i\boldsymbol{Q}_i^T\left\{\underset{1\le j\le n_i}{\text{stack}}(\boldsymbol{d}_{2ij})\right\} - \boldsymbol{C}_{1i}^T\boldsymbol{c}_{1i}\right]\right) \\
&= \boldsymbol{A}^{11}\left(\sum_{i=1}^{m}\left[\begin{bmatrix}\boldsymbol{C}_{1i}\\ \boldsymbol{C}_{2i}\end{bmatrix}^T\begin{bmatrix}\boldsymbol{c}_{1i}\\ \boldsymbol{c}_{2i}\end{bmatrix} - \boldsymbol{C}_{1i}^T\boldsymbol{c}_{1i}\right]\right) \\
&= \boldsymbol{A}^{11}\left\{\sum_{i=1}^{m}(\boldsymbol{C}_{1i}^T\boldsymbol{c}_{1i} + \boldsymbol{C}_{2i}^T\boldsymbol{c}_{2i} - \boldsymbol{C}_{1i}^T\boldsymbol{c}_{1i})\right\} \\
&= \boldsymbol{A}^{11}\sum_{i=1}^{m}\boldsymbol{C}_{2i}^T\boldsymbol{c}_{2i} \\
&= \boldsymbol{A}^{11}\left\{\underset{1\le i\le m}{\text{stack}}(\boldsymbol{C}_{2i})\right\}^T\left\{\underset{1\le i\le m}{\text{stack}}(\boldsymbol{c}_{2i})\right\} \\
&= \boldsymbol{A}^{11}\left(\boldsymbol{Q}\begin{bmatrix}\boldsymbol{R}\\ \boldsymbol{0}\end{bmatrix}\right)^T\left\{\underset{1\le i\le m}{\text{stack}}(\boldsymbol{c}_{2i})\right\} \\
&= \boldsymbol{R}^{-1}\boldsymbol{R}^{-T}\begin{bmatrix}\boldsymbol{R}\\ \boldsymbol{0}\end{bmatrix}^T\boldsymbol{Q}^T\left\{\underset{1\le i\le m}{\text{stack}}(\boldsymbol{c}_{2i})\right\} \\
&= \boldsymbol{R}^{-1}\boldsymbol{R}^{-T}\boldsymbol{R}^T\boldsymbol{c} \\
&= \boldsymbol{R}^{-1}\boldsymbol{c},
\end{aligned}
$$

where the fourth equality holds by the orthogonality of $\boldsymbol{Q}_i$, the fifth equality holds by Step 1(b) and the ninth and eleventh equalities hold by Step 2 of Theorem 4. We get the other sub-vectors of $\boldsymbol{x}$ by a similar sequence of computations.

From Theorem 3 we have

$$|\boldsymbol{A}| = \left|\left(\boldsymbol{A}^{11}\right)^{-1}\right| \prod_{i=1}^{m}\left(\left|\boldsymbol{A}_{22,i} - \sum_{j=1}^{n_i} \boldsymbol{A}_{12,\,i,\,j}\,\boldsymbol{A}_{22,ij}^{-1}\,\boldsymbol{A}_{12,\,i,\,j}^{T}\right| \prod_{j=1}^{n_i} |\boldsymbol{A}_{22,ij}|\right).$$

From part 3. of Theorem 4 we have $\boldsymbol{A}^{11} = \boldsymbol{R}^{-1}\boldsymbol{R}^{-T}$ so steps similar to those used for simplication of $|\boldsymbol{A}|$ in the proof of Theorem 2 lead to

$$\left|\left(\boldsymbol{A}^{11}\right)^{-1}\right| = \big(\text{product of the diagonal entries of } \boldsymbol{R}\big)^2.$$

From the above arguments involving the $\boldsymbol{H}_{22,i}$ matrices,

$$\begin{aligned}\left|\boldsymbol{A}_{22,i} - \sum_{j=1}^{n_i} \boldsymbol{A}_{12,\,i,\,j}\,\boldsymbol{A}_{22,ij}^{-1}\,\boldsymbol{A}_{12,\,i,\,j}^{T}\right| &= |\boldsymbol{H}_{22,i}| = |\boldsymbol{R}_i^T\boldsymbol{R}_i| = \big(|\boldsymbol{R}_i|\big)^2 \\ &= \big(\text{product of the diagonal entries of } \boldsymbol{R}_i\big)^2.\end{aligned}$$

for $1 \le i \le m$. Lastly,

$$\big|\boldsymbol{A}_{22,ij}\big| = |\ddot{\boldsymbol{B}}_{ij}^T\ddot{\boldsymbol{B}}_{ij}| = |\boldsymbol{R}_{ij}^T\boldsymbol{R}_{ij}| = \big(|\boldsymbol{R}_{ij}|\big)^2 = \big(\text{product of the diagonal entries of } \boldsymbol{R}_{ij}\big)^2.$$

for $1 \le i \le m,\ 1 \le j \le n_i$. The stated result for $|\boldsymbol{A}|$ immediately follows.

Reference

Harville, D.A. (2008). *Matrix Algebra from a Statistician's Perspective*. New York: Springer.